THE PYTHAGOREAN PLATO AND THE GOLDEN SECTION: A STUDY IN ABDUCTIVE INFERENCE

BY

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A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1983

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By

Scott A. Olsen

This dissertation is respectfully dedicated to two excellent teachers, the late Henry Mehlberg, and Dan Pedoe. Henry Mehlberg set an impeccable example in the quest for knowledge. And Dan Pedoe instilled in me a love for the ancient geometry.

ACKNOWLEDGMENTS

I would like to thank my Committee members Dr. Ellen Haring, Dr. Thomas W. Simon, Dr. Robert D'Amico, and Dr. Philip Callahan, and outside reader Dr. Joe Rosenshein for their support and attendance at my defense. I would also like to thank Karin Esser and Jean Pileggi for their help in this endeavour.

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Abstract of Dissertation Presented to the Graduate Council of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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AUGUST 1983

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The thesis of this dissertation is an interweaving relation of three factors. First is the contention that Plato employed and taught a method of logical discovery, or analysis, long before Charles Sanders Peirce rediscovered the fundamental mechanics of the procedure, the latter naming it abduction. Second, Plato was in essential respects a follower of the Pythagorean mathematical tradition of philosophy. As such, he mirrored the secrecy of his predecessors by avoiding the use of explicit doctrinal writings. Rather, his manner was obstetric, expecting the readers of his dialogues to abduct the proper solutions to the problems and puzzles presented therein. Third, as a Pythagorean, he saw number, ratio, and

proportion as the essential underlying nature of things. In particular he saw the role of the golden section as fundamental in the structure and aesthetics of the Cosmos.

Plato was much more strongly influenced by the Pythagoreans than is generally acknowledged by modern scholars. The evidence of the mathematical nature of his unwritten lectures, his disparagement of written doctrine, the mathematical nature of the work in the Academy, the mathematical hints embedded in the "divided line" and the Timaeus, and Aristotle's references to a doctrine of mathematicals intermediate between the Forms and sensible things, tend to bear this out. In his method of analysis, Plato would reason backwards to a hypothesis which would explain an anomalous phenomenon or theoretical dilemma. In many ways Plato penetrated deeper into the mystery of numbers than anyone since his time. This dissertation is intended to direct attention to Plato's unwritten doctrines, which centered around the use of analysis to divine the mathematical nature of the Cosmos.

CHAPTER I

INTRODUCTION

The thesis of this dissertation is an interweaving relation of three factors. First is the contention that Plato employed and taught a method of logical discovery long before Charles Sanders Peirce rediscovered the fundamental mechanics of this procedure, the latter naming it abduction. Second, Plato was in essential respects a follower of the Pythagorean mathematical tradition of philosophy. As such he mirrored the secrecy of his predecessors by avoiding the use of explicit doctrinal writings. Rather, his manner was obstetric, expecting the readers of his dialogues to abduct the proper solutions to the problems he presented. Third, as a Pythagorean he saw number, ratio, and proportion as the essential underlying nature of things. Both epistemologically and ontologically, number is the primary feature of his philosophy. Through an understanding of his intermediate doctrine of mathematicals and the soul, it will be argued that Plato saw number, ratio, and proportion literally infused into the world. The knowledge of man and an appreciation of what elements populate the Cosmos for Plato depends upon this apprehension of number in things. And in

particular it involves the understanding of a particular ratio, the golden section (tome)[1], which acted as a fundamental modular in terms of the construction and relation of things within the Cosmos.

Several subsidiary issues will emerge as I proceed through the argument. I will list some of these at the outset so that the reader may have a better idea of where my argument is leading. One feature of my positon is that, though not explicitly exposing his doctrine in the dialogues, Plato nevertheless retained a consistent view throughout his life regarding the Forms and their mathematical nature. The reason there is confusion about Plato's mathematical doctrine of Number-Ideas and mathematicals is because commentators have had a hard time tallying what Aristotle has to say about Plato's doctrine with what appears on the surface in Plato's dialogues. The problem is compounded due to the fact that, besides not explicitly writing on his number doctrine, Plato's emphasis is on midwifery throughout his works. In the early so-called Socratic dialogues the reader is left confused because no essential definitions are fastened upon. However, the method of cross-examination (elenchus)[2] as an initial stage of dialectical inquiry is employed to its fullest. Nevertheless, the middle dialogues quite literally expose some of the mathematical doctrine for those who have eyes to see it. But the reader must employ abduction, reasoning backwards from the puzzles, problems,

and hints to a suitable explanatory hypothesis. This abductive requirement is even more evident in the later dialogues, especially the Theaetetus, Parmenides, and Sophist.[3] There, if one accepts the arguments on their surface, it appears that Plato is attacking what he has suggested earlier regarding knowledge and the Forms. But this is not the case.

A further perplexing problem for many scholars enters the picture when one considers what Aristotle has to say about Plato's unwritten teachings. It has led to the mistaken view that Plato changed his philosophy radically in later life. However, my contention is that a careful reading of what Aristotle has to say upon the matter helps to unfold the real underlying nature of the dialogues. Mathematical concepts are present in one way or another throughout the dialogues. Possibly obscure at the beginning, they become central in the middle dialogues, especially the Republic. And the later Philebus and Epinomis attest to the retention of the doctrine. Copleston, who I am in agreement with on this matter, summed up the position as follows:

> There is indeed plenty of evidence that Plato continued to occupy himself throughout his years of academic and literary activity with problems arising from the theory of Forms, but there is no real evidence that he ever radically changed his doctrine, still less that he abandoned it altogether. . . . It has sometimes been asserted that the mathematisation of the Forms, which is ascribed to Plato by Aristotle, was a doctrine of Plato's old age, a relapse into Pythagorean "mysticism," but Aristotle does

> not say that Plato changed his doctrine, and the only reasonable conclusion to be drawn from Aristotle's words would appear to be that Plato held more or less the same doctrine, at least during the time that Aristotle worked under him in the Academy. (Copleston, 1962, p. 188)

Others, like Cherniss, get around the problem by accusing Aristotle of "misinterpreting and misrepresenting" Plato (Cherniss, 1945, p. 25). This is ludicrous. We need only recall that Aristotle was in the Academy with Plato (until the master's death) for 19 (possibly 20) years. Surely he should know quite well what Plato had to say. Fortunately we have some record of what Plato had to say in his unwritten lectures. This helps to fill the gap. But unfortunately the remnant fragments are sparse, though very telling. The view of Cherniss' only indicates the extreme to which some scholars will move in an attempt to overcome the apparent disparity. As Copleston goes on to say,

> . . . though Plato continued to maintain the doctrine of Ideas, and though he sought to clarify his meaning and the ontological and logical implications of his thought, it does not follow that we can always grasp what he actually meant. It is greatly to be regretted that we have no adequate record of his lectures in the Academy, since this would doubtless throw great light on the interpretation of his theories as put forward in the dialogues, besides conferring on us the inestimable benefit of knowing what Plato's "real" opinions were, the opinions that he transmitted only through oral teaching and never published. (Copleston, 1962, pp. 188-189)

My own view on the matter is that if we look closely enough at the extant fragments, in conjunction with the Pythagorean background of Plato's thought, and using these

as keys, we can unlock some of the underlying features of Plato's dialogues. But this is premised on the assumption that Plato is in fact being obstetric in the dialogues. I will argue that this is the case, and that further, sufficient clues are available to evolve an adequate reconstruction of his mathematical-philosophical doctrine.

Thus, a central theme running throughout this dissertation is that the words of Aristotle will help to clarify the position of Plato. Rather than disregard Aristotle's comments, I will emphasize them. In this way I hope to accurately explicate some of the features of Plato's mathematical doctrine and the method of discovery by analysis that he employed. Plato deserves an even richer foundation in the philosophies of science and logic than he has heretofore been credited with. His method of analysis, of the upward path of reasoning backwards from conclusion to premises (or from facts to hypothesis or principle), lies at the very roots of scientific discovery. The mistaken view of a strictly bifurcated Platonic Cosmos, with utter disdain for the sensible world, has done unjust damage to the reputation of Plato among those in science. This is unfortunate and needs to be remedied.

I set for myself the following problem at the outset. When we arrive at the Timaeus we will see how the elements, the regular solids, are said to be constructed out of two kinds of right-angled triangles, one isosceles and the other scalene. But he goes on to say,

> These then . . . we assume to be the original elements of fire and other bodies, but the priciples which are prior to to these deity only knows, and he of men who is a friend of deity. (Timaeus 53d-e)

I contend that this is a cryptic passage designed by the midwife Plato to evoke in the reader a desire to search for the underlying Pythagorean doctrine. To some it conceals the doctrine. However, to others it is intended to reveal, if only one is willing to reason backwards to something more primitive. Thus Plato goes on to say,

> . . . anyone who can point out a more beautiful form than ours for the construction of these bodies shall carry off the palm, not as an enemy, but as a friend. Now, the one which we maintain to be the most beautiful of all the many triangles . . . is that of which the double forms a third triangle which is equilateral. The reason of this would be too long to tell; he who disproves what we are saying, and shows that we are mistaken, may claim a friendly victory. (Timaeus 54a-b)

This is the problem: what more beautiful or primitive form could there be for the construction of these bodies?

My views are undeniably in the vein of the Neopythagorean and Neoplatonic traditions. But my contention is that it is to the Pythagorean Neoplatonists that we must turn if we are to truly understand Plato. I have found a much greater degree of insight into Plato in the Neopythagoreans and Neoplatonists than in the detailed work of the logic choppers and word mongers. As Blavatsky once said regarding one of the Neoplatonists, Thomas Taylor, the English Platonist,

> the answer given by one of Thomas Taylor's admirers to those scholars who criticized his translations of Plato [was]: "Taylor might have known less Greek than his critics, but he knew more Plato." (Blavatsky, 1971, vol. 2, p. 172)

As Flew has written, the origins of the Neoplatonic interpretation

> go back to Plato's own lifetime. Its starting-point was Plato's contrast between eternal Ideas and the transient objects of sense, a contrast suggesting two lines of speculative enquiry. First, what is the connection, or is there anything to mediate between intelligibles and sensibles, the worlds of Being and of Becoming? Second, is there any principle beyond the Ideas, or are they the ultimate reality? (Flew, 1979, p. 254)

This dissertation speaks directly to the former question, although, I will have something to say about the latter as well.

Notes

[1] Most subsequent Greek words will be transliterated. Although I will occasionally give the word in the original Greek. The golden section, tome, was often referred to by the Greeks as division in mean and extreme ratio.

[2] Cross-examination, or elenchus, is an important stage in the dialectical ascent to knowledge. It is employed to purge one of false beliefs. Through interrogation one is led to the assertion of contradictory beliefs. This method is decidedly socratic. Plato emphasized a more cooperative effort with his students in the Academy.

[3] All citations to works of Plato are according to the convention of dialogue and passage number. All citations are to H. Cairns and E. Hamilton, eds., 1971, The Collected Dialogues of Plato, Princeton: Princeton University Press. The one major exception is that all Republic quotes are from D. Lee, transl., 1974, The Republic, London: Penquin.

CHAPTER II

ABDUCTION

Peirce

I choose to begin with Charles Sanders Peirce, because better than anyone else he seems to have grasped the significance of the logic of backwards reasoning, or abduction. Once the position of Peirce is set out, with some explicit examples, I will return to the examination of Plato's philosophy.

What Peirce termed abduction (or alternatively, reduction, retroduction, presumption, hypothesis, or novel reasoning) is essentially a process of reasoning backwards from an anomalous phenomenon to a hypothesis which would adequately explain and predict the existence of the phenomenon in question. It lies at the center of the creative discovery process. Abduction occurs whenever our observations lead to perplexity. Abduction is the initial grasping at explanation. It is a process by which one normalizes that which was previously anomalous or surprising. Peirce's basic formula is very simple:

> The surprising fact, C, is observed; But if A were true, C would be a matter of course, Hence, there is reason to suspect that A is true. (Peirce 5.189)[1]

Abduction follows upon the initiation of a problem or puzzling occurrence. The great positive feature about abduction is that it can lead to very rapid solutions. On this view discovery takes place through a series of leaps, rather than a gradual series of developments. As Peirce said, gradual progression

> is not the way in which science mainly progresses. It advances by leaps; and the impulse for each leap is either some new observational resource, or some novel way of reasoning about the observations. Such novel way of reasoning might, perhaps, be considered as a new observational means, since it draws attention to relations between facts which should previously have been passed by unperceived. (Buchler, 1955, p. 51)

Whenever present theories cannot adequately explain a fact, then the door for abduction opens. Most major scientific discoveries can be correctly viewed as an abductive response to perplexing, or anomalous, phenomena. Thus, Einstein was struck by certain perplexing, seemingly unaccountable features of the world. Sometimes these anomalous features cluster about a particular problem. When this occurs, and is perceived by the individual, great creative abductive solutions become possible. Thus, as Kuhn points out,

> Einstein wrote that before he had any substitute for classical mechanics, he could see the interrelation between the known anomalies of black-body radiation, the photoelectric effect and specific heats. (Kuhn, 1970, p. 89)

Thus, on the basis of these perplexing facts, Einstein was

able to reason backwards to a suitable hypothesis that would reconcile and adapt each of these puzzling features.

Abduction is the process by which surprising facts invoke an explanatory hypothesis to account for them. Thus, abduction "consists in studying facts and devising a theory to explain them" (Peirce 5.145). It "consists in examining a mass of facts and in allowing these facts to suggest a theory" (Peirce 8.209). And it is "the logic by which we get new ideas" (Peirce 7.98).

For example, Maslow was doing abduction when he surveyed the data and observations, and, reasoning backwards, inferred the hypothesis of self-actualization. The fact that he called it "partly deductive," not knowing the correct label, does not affect the nature of his abduction.

> I have published in another place a survey of all the evidence that forces us in the direction of a concept of healthy growth or of self-actualizing tendencies. This is partly deductive evidence in the sense of pointing out that unless we postulate such a concept, much of human behavior makes no sense. This is on the same scientific principle that led to the discovery of a hitherto unseen planet that had to be there in order to make sense of a lot of other observed data. (Maslow, 1962, pp. 146-147)

Abduction is to be clearly distinguished from deduction and induction. Nevertheless, the three logical methods are mutually complementary. However, deduction only follows upon the initial abductive grasping of the new hypothesis. The deductive consequences or predictions are then set out. Induction then consists of the experimental

testing and observation to see if in fact the consequences deduced from the new hypothesis are correct. If not, then abduction begins again seeking a new or modified hypothesis which will more adequately explain and predict the nature of our observations.

Thus, as Peirce points out,

> abduction is the process of forming an explanatory hypothesis. It is the only logical operation which introduces any new ideas; for induction does nothing but determine a value, and deduction merely evolves the necessary consequences of a pure hypothesis. (Peirce 5.171)

Abduction does not have the nature of validity that, for example, deduction possesses. Abduction is actually a form of the so-called fallacy of affirming the consequent. The abducted hypothesis cannot in any way be apprehended as necessary. It must be viewed as a tentative conjecture, and at best may be viewed as likely. When we say, the surprising fact C is observed, but if A were true, C would follow as a matter of course, the very most that we can do is say that we therefore have reason to suspect A. Abduction is a logical method of hypothesis selection, and it is extremely effective, especially when anomalies are used to guide one toward the best explanation. But abductions may turn out to have false results.

> The function of hypothesis [abduction] is to substitute for a great series of predicates forming no unity in themselves, a single one (or small number) which involves them all, together (perhaps) with an indefinite number of others. It is, therefore, also a reduction of a manifold to a unity. Every deductive syllogism may be put into the form:

> If A, then B; But A: Therefore, B. And as the minor premiss in this form appears as antecedent or reason of a hypothetical proposition, hypothetic inference [abduction] may be called reasoning from consequent to antecedent. (Peirce 5.276)

This notion of reasoning backwards from consequent to hypothesis is central to abduction. One thing it shares in common with induction is that it is "rowing up the current of deductive sequence" (Peirce, 1968, p. 133). But abduction and induction are to be clearly distinguished. Abduction is the first step of explanatory discovery, the grasping of the hypothesis or account. Induction is the testing of the hypothesis that follows the previous abductive hypothesis selection and the deductive prediction

> of consequences. The operation of testing a hypothesis by experiment, which consists in remarking that, if it is true, observations made under certain conditions ought to have certain results, and noting the results, and if they are favourable, extending a certian confidence to the hypothesis, I call induction. (Buchler, 1955, p. 152)

There is also a sense in which abduction and induction can be contrasted as opposing methods.

> The induction adds nothing. At the very most it corrects the value of a ratio or slightly modifies a hypothesis in a way which had already been contemplated as possible. Abduction, on the other hand, is merely preparatory. It is the first step of scientific reasoning, as induction is the concluding step. They are the opposite poles of reason. . . . The method of either is the very reverse of the other's. Abduction seeks a theory. Induction seeks the facts. (Peirce 7.127-7.218)

The important point here is that abduction occurs in the process of discovery. It is distinct from the later

process of justification of the hypothesis. Hanson made this point strongly when he stated that

> the salient distinction of "The Logic of Discovery" consisted in separating (1) Reasons for accepting a hypothesis, H, from (2) Reasons for suggesting H in the first place. (Hanson, 1960, p. 183)

Abduction, the logic of discovery, underlies the latter above. The robust anomaly R provides reasons to suspect hypothesis H is true. This is the case, simply because if H were true, then R would follow as a matter of course. Hence we have reasons for suggesting or selecting H.

Abduction is the mark of the great theoretical scientists. Through contemplation of the observables, especially the puzzling observables, the theoretician reasoning backwards fastens upon a hypothesis adequate to explain and predict the occurrence of the anomalies. This, in effect, defuses the anomalous nature of the observables, having the effect of normalizing them.

A crucial feature of the activity of abduction is the role that is played by the anomaly, R. It has the function of directing one to the type of hypothesis that is required. As Hanson pointed out, "to a marked degree [the] observations locate the type of hypothesis which it will be reasonable ultimately to propose" (Hanson, 1960, p. 185).

The overall interplay of abduction, deduction, and induction can be appreciated more fully when considering the following passage by Peirce:

> The Deductions which we base upon the hypothesis which has resulted from Abduction

> produce conditional predictions concerning our future experience. That is to say, we infer by Deduction that if the hypothesis be true, any future phenomena of certain descriptions must present such and such characters. We now institute a course of quasi-experimentation in order to bring these predictions to the test, and thus to form our final estimate of the value of the hypothesis, and this whole proceeding I term Induction. (Peirce 7.115, fn.27)

Abduction, or the logic of discovery, has unfortunately been ignored for some time. As Paul Weiss says, "it is regrettable that the logicians are not yet ready to follow Peirce into this most promising field [abduction]" (Bernstein, 1965, p. 125). Only recently has there been a rebirth of interest.

> In the case of abduction, Peirce singles out as an independent form of inference the formulation of hypotheses for inductive testing. All this is well known, but, we fear, too much ignored outside the constricted space of Peirce scholarship. Unfortunately, the notion of abductive inference, which is peculiarly Peirce's, has not exerted an influence proportionate to the significance of its insight. (Harris & Hoover, 1980, p. 329)

The only point where Harris and Hoover err in the prior statement is in attributing abduction as solely belonging to Peirce. But this is a mistake. Peirce himself acknowledged his Greek sources of the abductive logic. I will argue that the roots of abduction lie in Plato, and his work in the Academy, and his Pythagorean predecessors. But first we will consider some of the more recent developments of Peircean abduction, and then some actual historical examples.

Consider the following example,

> I catch the glint of light on metal through the trees by the drive, remark that I see the family car is there, and go on to infer my son is home. It may be said that taken literally I have misdescribed things. What I see, it may be said, is a flash of light through the trees. Strictly I infer, but do not see, that the car is there. . . . I reason backward from what I see, the flash of light on metal, and my seeing it, to a cause the presence of which I believe to be sufficient to explain my experience. Knowing the situation, and knowing the way things look in curcumstances like these, I infer that the car is in the drive. (Clark, 1982, pp. 1-2)

Clark goes on to describe the argument form involved. Let q be the puzzling perceptual occurrence or anomaly. In this case it was the glint of light passing through the trees. Let p stand for the car is in the drive. Let B stand for the belief that if p (the car is in the drive), and other things being equal, then q (the glint of light) would occur. We can then reconstruct the argument as follows.

1. q (puzzling glint of light),
2. But B (belief that $p \supset q$),
3. Therefore, p (car in drive).

> I conclude from my premises, q and B, that [hypothesis] p. I conclude that the family car is there, this being the hypothesis I draw the truth of which I believe is sufficient to account for that puzzling perceptual happening, q. (Clark, 1982, p. 2)

But if this is abduction, are we not simply employing the fallacy of affirming the consequent? Or is it something more?

> This pattern of reasoning is quite common. And it is after all a sort of reasoning. There is here a texture of structured thoughts leading to a conclusion. Moreover, there's something sensible about it. It is not just silly. But of course reasoning this way, I have sinned deductively. My reasoning is not deductively valid. (q and B might after all quite well be true and yet [hypothesis p] false. Perhaps it is not in fact the car but a visiting neighbor's camper whose flash of light on metal I catch). Peirce insisted that all creativity has its source in sin: reasoning of this general sort is the only creative form of inference. It is the only sort that yields as conclusions new hypotheses not covertly asserted in the premises; new hypotheses now to be tested and examined; hypotheses which may determine whole new lines of inquiry. This reasoning is, he thought, quite ubiquitous, present indeed in all perception but in nearly every area of contingent inquiry as well. (It is philosophical commonplace, too. How frequently we reason backward from an epistemological puzzle to an ontological posit.) Peirce, in characterizing this backward, abductive, reasoning which runs from effects to hypotheses about causes sufficient to ensure them, has implicitly answered the title question. When is a fallacy valid? Answer: When it is a good abduction. (Clark, 1982, p. 2)

Clark proceeds admirably, struggling with abduction, attempting to define its formal standards for validity and soundness. "It is . . . the need to characterize abductive soundness which forces the nontrivial nature of abduction on us"(Clark, 1982, p. 3). In the very process of this attempt, Clark has reasoned abductively. In a very analagous manner, this dissertation is an exercise in abduction, reasoning backwards from the Platonic puzzles (i.e., the dialogues and the extraneous statements

regarding Plato's doctrines), to an explanatory hypothesis regarding them.

> As an especial philosophical application and final example, it is perhaps worth remarking that this account of the nature of abduction is itself an exercise in abduction. We have reasoned backward from a puzzling fact--the widespread employment in philosophical inquiry of arguments which are deductively fallacious--to an attempt to characterize an adequate explanation of the phenomenon. We have tried to sketch minimal formal standards by which abductions can be evaluated as valid or sound, and their employment justified. I wish I could say more about what is important about abduction and the competition of sufficient hypotheses. I wish I could formulate an articulate formal system of abduction. But even a sketch like this is something. It seems to me at least to override an obvious competitor to explaining our ubiquitous use of these forms of inference; the view that these are just logical lapses--irrational applications of the fallacy of asserting the consequent. (Clark, 1982, p. 12)

Eratosthenes & Kepler

The great discovery of Eratosthenes, the Librarian at Alexandria, provides a good example of abduction. He pondered over the puzzling fact that on the summer solstice at noonday the sun was at its zenith directly overhead in Syene, Egypt, and yet 500 miles north at that precise moment in Alexandria, the sun was not directly at its zenith. He abducted that this must be due to the curvature of the earth away from the sun. He went further and reasoned that he could determine the amount of curvature of the earth through geometrical calculation by measuring the length of shadow cast at Alexandria at noon on the summer

solstice. Knowing the distance from Syene to Alexandria, he was then able to quite accurately (circa 240B.C.) calculate the diameter and circumference of the earth.

> Eratosthenes worked out the answer (in Greek units), and, as nearly as we can judge, his figures in our units came out at about 8,000 miles for the diameter and 25,000 miles for the circumference of the earth. This, as it happens, is just about right. (Asimov, 1975, vol. 1, p. 22)

In view of the perplexing difference in the position of the sun in the two cities on the summer solstice, Eratosthenes was able to reason backwards to a hypothesis, i.e., the earth is round and therefore curves away from the rays of the sun, which would render that anomalous phenomenon the expected.

Kepler[2] is another example of brilliant abductive inferences. Both Peirce and Hanson revere the work of Kepler. Hanson asks,

> was Kepler's struggle up from Tycho's data to the proposal of the elliptical orbit hypothesis really inferential at all? He wrote De Motibus Stellae Martis in order to set out his reason for suggesting the ellipse. These were not deductive reasons; he was working from explicanda to explicans [reasoning backwards]. But neither were they inductive--not, at least, in any form advocated by the empiricists, statisticians and probability theorists who have written on induction. (Hanson, 1972, p. 85)

The scientific process of discovery may at times be viewed as a series of explanatory approximations to the observed facts. An abductively conjectured hypothesis will often approximate to an adequate explanation of the facts. One continues to attempt to abduct a more complete

hypothesis which more adequately explains the recalcitrant facts. Hence there will occassionally occur the unfolding of a series of hypotheses. Each hypothesis presumably approximates more closely to an adequate explanation of the observed facts. This was the case with Kepler's work, _De Motibus Stellae Martis._ As Peirce points out,

> . . . at each stage of his long investigation, Kepler has a theory which is approximately true, since it approximately satisfies the observations . . . and he proceeds to modify this theory, after the most careful and judicious reflection, in such a way as to render it more rational or closer to the observed fact. (Buchler, 1955, p. 155)

Although abduction does involve an element of guess-work, nevertheless, it does not proceed capriciously.

> Never modifying his theory capriciously, but always with a sound and rational motive for just the modification he makes, it follows that when he finally reaches a modification--of most striking simplicity and rationality--which exactly satisfies the observations, it stands upon a totally different logical footing from what it would if it had been struck out at random. (Buchler, 1955, p. 155)

Hence, there is method to abduction. Rather than referring to it as a case of the fallacy of affirming the consequent, it would be better to term it directed affirmation of the consequent. The arrived at hypothesis will still be viewed as tentative. But as Peirce indicated there is a logical form to it.[3]

> Abduction, although it is very little hampered by logical rules, nevertheless is logical inference, asserting its conclusion only problematically, or conjecturally, it is

> true, but nevertheless having a perfectly definite logical form. (Peirce 5.188)

A crucial feature of abduction is that it is originary in the sense of starting a new idea. It inclines, rather than compels, one toward a new hypothesis.

> At a certain stage of Kepler's eternal exemplar of scientific reasoning, he found that the observed longitudes of Mars, which he had long tried in vain to get fitted with an orbit, were (within the possible limits of error of the observations) such as they would be if Mars moved in an ellipse. The facts were thus, in so far, a likeness of those of motion in an elliptic orbit. Kepler did not conclude from this that the orbit really was an ellipse; but it did incline him to that idea so much as to decide him to undertake to ascertain whether virtual predictions about the latitudes and parallaxes based on this hypothesis would be verified or not. This probational adoption of the hypothesis was an abduction. An abduction is Originary in respect of being the only kind of argument which starts a new idea. (Buchler, 1955, p. 156)

A very simple way of expressing the anomalous orbit of Mars and the resulting abductive hypothesis is indicated by Hanson. It is relevant to note that it begins with an interrogation. "Why does Mars appear to accelerate at 90 [degrees] and 270 [degrees]? Because its orbit is elliptical" (Hanson, 1972, p. 87). Again putting the formula into its simplest form, we may say: the surprising fact R is observed, but what hypothesis H could be true that would make R follow as a matter of course? It is the upward reach for H that is fundamental to the notion of abduction.

Apagoge

How then is abduction related to Plato? The initial clue is given in a statement by Peirce.

> There are in science three fundamentally different kinds of reasoning, Deduction (called by Aristotle sunagoge or anagoge), Induction (Aristotle's and Plato's epagoge) and Retroduction [abduction] (Aristotle's apagoge). (Peirce 1.65)

Apagoge is defined as "I. a leading or dragging away. II. a taking home. III. payment of tribute. IV. as a law-term, a bringing before the magistrate" (Liddell and Scott, 1972, p. 76). There is thus the underlying notion of moving away from, or a return or reversion of direction.

Peirce's reference is to Aristotle's use of the term, apagoge. It is generally translated as reduction.

> By reduction we mean an argument in which the first term clearly belongs to the middle, but the relation of the middle to the last term is uncertain though equally or more probable than the conclusion; or again an argument in which the terms intermediate between the last term and the middle are few. For in any of these cases it turns out that we approach more nearly to knowledge. For example let A stand for what can be taught, B for knowledge, C for justice. Now it is clear that knowledge can be taught [AB]: but it is uncertain whether virtue is knowledge [BC]. If now the statement BC [virtue is knowledge] is equally or more probable than AC [virtue can be taught], we have a reduction: for we are nearer to knowledge, since we have taken a new term [B which gives premises AB and BC, on which the inquiry now turns], being so far without knowledge that A [what can be taught] belongs to C [virtue].[4] (<u>Prior Analytics</u> 69a20-30)

On this view then, reduction is the grasping of a new term which transforms the inquiry onto a new footing. According

to Aristotle we are nearer knowledge because by reducing the problem to something simpler, we are closer to solving it. By solving the new reduced problem, the solution to the original problem will follow.

The evidence is that the Aristotelian term apagoge has its roots in geometrical reduction. Thus Proclus says:

> Reduction is a transition from one problem or theorem to another, the solution or proof of which makes that which is propounded manifest also. For example, after the doubling of the cube had been investigated, they transformed the investigation into another upon which it follows, namely the finding of two means; and from that time forward they inquired how between two given straight lines two mean proportionals could be discovered. And they say that the first to effect the reduction of difficult constructions was Hippocrates of Chios, who also squared a lune and discovered many other things in geometry, being second to none in ingenuity as regards constructions. (Heath, 1956, vol. 1, p. 135)

Thus, we see the basic movement as later described by Peirce, in which a problem is solved or an anomaly explained, by the backwards reasoning movement to a hypothesis from which the anomalous phenomenon or solution would follow as a matter of course. The difference here is that in reduction there is an initial step toward arriving at a hypothesis from which the phenomenon or solution would follow, but the hypothesis is such that it still must be established. However, by selecting the hypothesis one has succeeded in reducing the problem to another, but simpler, problem. Hence, the Delian problem of doubling the cube was reduced to the problem of finding two mean proportionals between two given straight lines. As we

shall see subsequently, Archytas performed the initial step of reduction, and Eudoxus performed the final step of solution.

In a footnote to the Proclus passage above, Heath makes the following relevant remarks:

> This passage has frequently been taken as crediting Hippocrates with the discovery of the method of geomtrical reduction. . . . As Tannery remarks, if the particular reduction of the duplication problem to that to the two means is the first noted in history, it is difficult to suppose that it was really the first; for Hippocrates must have found instances of it in the Pythagorean geometry. . . . but, when Proclus speaks vaguely of "difficult constructions," he probably means to say simply that "this first recorded instance of a reduction of a difficult construction is attributed to Hippocrates." (Heath, 1956, vol.1, pp. 135-136)

This suggests that the real source of reduction or apagoge is the Pythagoreans. I will return to this point later.

It is also interesting to note that in the Proclus quotation above there is reference to the squaring of lunes. Aristotle, in the Prior Analytics passage cited above, goes on to refer to the squaring of the circle with the aid of lunules.

> Or again suppose that the terms intermediate between B [knowledge] and C [virtue] are few: for thus too we are nearer knowledge. For example let D stand for squaring, E for rectilinear figure, F for circle. If there were only one term intermediate between E [squaring] and F [circle] (viz. that the circle made equal to a rectilinear figure by the help of lunules), we should be near to knowledge. But when BC [virtue is knowledge] is not more probable than AC [virtue can be taught], and the intermediate terms are not few, I do not call this reduction: nor again when the statement BC [virtue is knowledge]

> is immediate: for such a statement is knowledge. (Prior Analytics 69a30-37)

My own view is that reduction as expressed by Aristotle is really a special limiting case of what Peirce termed abduction. There is a more general model of the abductive process available amongst the Greeks. And further, reduction does not precisely fit the basic formula Peirce has presented.

> The surprising fact, C, is observed; But if A were true, C would be a matter of course, Hence, there is reason to suspect that A is true. (Peirce 5.5189)

Reduction appears to be a species of this formula. However, there appears to be a more apt generic concept available amongst the Greeks. This I contend is the ancient method of analysis. In the end reduction may be seen to be closely allied to analysis. But successful analysis or abduction requires the discovery of an adequate hypothesis. It is possible that this is achieved through a series of reductions.

In reference to the discovery of lemmas, Proclus says,

> . . . certain methods have been handed down. The finest is the method which by means of analysis carries the thing sought up to an acknowledged principle, a method which Plato, as they say, communicated to Leodamas, and by which the latter, too, is said to have discovered many things in geometry. (Heath, 1956, vol. 1, p. 134)

Heath, in some insightful remarks, sees this analysis as similar to the dialectician's method of ascent. Thus he says:

> This passage and another from Diogenes

> Laertius to the effect that "He [Plato] explained (eisegasato) to Leodamos of Thasos the method of inquiry by analysis" have been commonly understood as ascribing to Plato the invention of the method of analysis; but Tannery points out forcibly how difficult it is to explain in what Plato's discovery could have consisted if analysis be taken in the sense attributed to it in Pappus, where we can see no more than a series of successive, reductions of a problem until it is finally reduced to a known problem. On the other hand, Proclus' words about carrying up the thing sought to an "acknowledged principle" suggest that what he had in mind was the process described at the end of Book VI of the Republic by which the dialectician (unlike the mathematician) uses hypotheses as stepping-stones up to a principle which is not hypothetical, and then is able to descend step by step verifying every one of the hypotheses by which he ascended. (Heath, 1956, vol. 1, p. 134, fn.1)

There is both some insight and some glossing over by Heath here. Heath is correct that there is a definite relation here between analysis and what Plato describes as the upward path in Book VI of the Republic. But he is mistaken when he tries to divorce Platonic analysis from mathematical analysis. They are very closely related. Part of the confusion stems from the fact that in the Republic Plato distinguishes the mathematician's acceptance of hypotheses and subsequent deductions flowing from them, from the hypotheses shattering upward ascent of the dialectician. However, the mathematician also employs the dialectical procedure when he employs reduction and mathematical analysis. In these instances, unlike his deductive descent, the mathematician reasons backwards (or upwards) to other hypotheses, from the truth of which the

solution of his original problem will follow. This basic process is common to both mathematician and dialectician. The common denominator is the process of reasoning backwards.

A further problem resulting in the confusion is that it is not clear what is meant by references to an ancient method of analysis. Heath is perplexed as well. Thus he writes,

> It will be seen from the note on Eucl. XIII. 1 that the MSS. of the Elements contain definitions of Analysis and Synthesis followed by alternative proofs of XIII. 1-5 after that method. The definitions and alternative proofs are interpolated, but they have great historical interest because of the possibility that they represent an ancient method of dealing with propositions, anterior to Euclid. The propositions give properties of a line cut "in extreme and mean ratio," and they are preliminary to the construction and comparison of the five regular solids. Now Pappus, in the section of his Collection [Treasury of Analysis] dealing with the latter subject, says that he will give the comparisons between the five figures, the pyramid, cube, octahedron, dodecahedron and icosahedron, which have equal surfaces, "not by means of the so-called analytical inquiry, by which some of the ancients worked out the proofs, but by the synthetical method." The conjecture of Bretschneider that the matter interpolated in Eucl. XIII is a survival of investigations due to Eudoxus has at first sight much to commend it. In the first place, we are told by Proclus that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section, and employed in them the method of analysis." (Heath, 1956, vol. 1, p. 137)

This is an extremely interesting passage. Is this the same method of analysis that was earlier attributed to the discovery of Plato? However, if the method is ancient,

then at best Plato could only have discovered it in the work of his predecessors, presumably the Pythagoreans. And what about "the section" (tome), and the theorems that Plato originated (and Eudoxus extended) regarding it?

> It is obvious that "the section" was some particular section which by the time of Plato had assumed great importance; and the one section of which this can safely be said is that which was called the "golden section," namely the division of a straight line in extreme and mean ratio which appears in Eucl. II. 11 and is therefore most probably Pythagorean. (Heath, 1956, vol. 1, p. 137)

If Plato had done so much work on this Pythagorean subject, the golden section, and further, his pupil Eudoxus was busy developing theorems regarding it, and further, both were using a method of analysis that may have ancient Pythagorean origins as well, then why is there no straightforward mention of this in the dialogues? Could the actual practice of what was occurring within the Academy have been so far removed from what is in the dialogues? Why was there such a discrepancy between practice and dialogue? These are some of the questions that will be answered in the course of this dissertation.

Focussing upon the question of analysis for the moment, there are interpolated definitions of analysis and synthesis in Book XIII of Euclid's <u>Elements</u>. Regarding the language employed, Heath says that it "is by no means clear and has, at the best, to be filled out" (Heath, 1956, vol. 1, p. 138).

> Analysis is an assumption of that which is sought as if it were admitted [and the

> passage] through its consequences [antecedents] to something admitted (to be) true. Synthesis is an assumption of that which is admitted [and the passage] through its consequences to the finishing or attainment of what is sought. (Heath, 1956, vol. 1, p. 138)

Unfortunately this passage is quite obscure. Fortunately Pappus has preserved a fuller account. However, it too is a difficult passage. One might even speculate as to whether these passages have been purposefully distorted.

> The so-called Treasury of Analysis is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements [i.e., Euclid's], are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines . . . and proceeds by way of analysis and synthesis. Analysis then takes that which is sought as if it were admitted and passes from it through its successive consequences[5] [antecedents] to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were (already) done (gegonos), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (anapalin lusin). (Heath, 1956, vol. 1, p. 138)

Thomas translates the last two words of the former passage, anapalin lusin, as "reverse solution"[6] (Thomas, 1957, vol. 2, p. 597). It is this "solution backwards" or "reverse solution" that I contend lay at the center of Plato's dialectical method.

Cornford is one of the few commentators to have any real insight into the passage from Pappus.

> . . . modern historians of mathematics--"careful studies" by Hankel, Duhamel, and Zeuthen, and others by Ofterdinger and Cantor--have made nonsense of much of it by misunderstanding the phrase "the succession of sequent steps" (τῶν ἑξῆς ἀκολουθων) as meaning logical "consequences," as if it were τὰ συμβαίνοντα . Some may have been misled by Gerhardt (Pappus, vii, viii, Halle, 1871), who renders it "Folgerungen." They have been at great pains to show how the premisses of a demonstration can be the consequences of the conclusion. The whole is clear when we see--what Pappus says--that the same sequence of steps is followed in both processes--upwards in Analysis, from the consequence to premisses implied in that consequence, and downwards in synthesis, when the steps are reversed to frame the theorem or demonstrate the construction "in the natural (logical) order." You cannot follow the same series of steps first one way, then the opposite way, and arrive at logical consequences in both directions. And Pappus never said you could. He added ἑξῆς to indicate that the steps "follow in succession "but are not, as ἀκόλουθα alone would suggest, logically "consequent" in the upward direction. (Cornford, 1965, p. 72, fn.1)

On the "abduction" view I am maintaining, Cornford has hit upon an acceptable interpretation of the Pappus passage. It is this reverse inference from conclusion to premise that was at the center of Plato's method of discovery. As Cornford goes on to say,

> Plato realized that the mind must possess the power of taking a step or leap upwards from the conclusion to the premiss implied in it. The prior truth cannot, of course, be deduced or proved from the conclusion; it must be grasped (ἅψασθαι , <u>Republic</u> 511b) by an act of analytical penetration. Such an act is involved in the solution "by way of

> hypothesis" at Meno 86. . . . The geometer directly perceives, without discursive argument, that a prior condition must be satisfied if the desired construction is to follow. (Cornford, 1965, p. 67)

But Cornford is not without opposition to his account. Robinson takes direct issue with him on the matter, claiming that all historians of Greek mathematics agree with the non-Cornford interpretation.

> The historians of Greek mathematics are at one about the method that the Greek geometers called analysis. Professor Cornford, however, has recently rejected their account and offered a new one. . . . Professor Cornford is mistaken and the usual view correct. (Robinson, 1969, p. 1)

But if this is true, then why is there such a mystery around the interpretation of the Pappus passage? And why the mystery surrounding what was meant by Plato's discovery of analysis? My view is that Cornford has gone far in uncovering part of an old mystery about analysis. The Cornford interpretation clearly lends support to the abduction view of Plato's method that I am advocating.

Actually, the medieval philosopher John Scotus Eriugena captured some of the underlying meaning of analysis when he distinguished the upward and downward movements of dialectic. In the Dialectic of Nature, he refers to this dual aspect of dialectic

> which divides genera into species and resolves species into genera once more. . . . There is no rational division . . . which cannot be retraced through the same set of steps by which unity was diversified until one arrives again at that initial unit which remains inseperable in itself. . . . Analytic comes from the verb analyo meaning "I return"

> or "I am dissolved." From this the term analysis is derived. It too can be translated "dissolution" or "return," but properly speaking, analysis refers to the solution of questions that have been proposed, whereas analytic refers to the retracing of the divisions of forms back to the source of their division. For all division, which was called "merismos" by the Greeks', can be viewed as a downward descent from a certain definite unit to an indefinite number of things, that is, it proceeds from the most general towards the most special. But all recollecting, as it were is a return again and this begins from the most special and moves towards the most general. Consequently, there is a "return" or "resolution" of individuals into forms, forms into genera. . . . (Whippel & Wolter, 1969, pp. 116-117)

Now the earlier remark cited by Heath (supra. pp.24-25), though missing the mark, may be insightful as to what analysis is.

> Tannery points out forcibly how difficult it is to explain in what Plato's discovery could have consisted if analysis be taken in the sense attributed to it in Pappus, where we see no more than a series of successive, reductions of a problem until it is finally reduced to a known problem. (Heath, 1956, vol. 1, p. 134, fn.1)

But this "series of reductions" may be fundamentally what was involved. Knowledge would be arrived at through a series of apagoges. Plato may have discovered unique uses of analysis, that extended beyond his predecessors. It may be that he discovered that analysis, or backward reasoning, may apply to propositions other than mathematical.[7] On the other hand, he may have simply "discovered" this more esoteric technique in the ancient geometrical tradition of the Pythagoreans. However, its true significance should be

considered within the context in which it is openly stated it was used. That is, in particular it should be considered in terms of what is said about Plato and Eudoxus as to the theorems regarding the section, and their discovery by analysis.

> As Cantor points out, Eudoxus was the founder of the theory of proportions in the form in which we find it in Euclid V., VI., and it was no doubt through meeting, in the course of his investigations, with proportions not expressible by whole numbers that he came to realise the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes. The "golden section" would furnish such a case. And it is even mentioned by Proclus in this connexion. He is explaining that it is only in aritmetic that all quantities bear "rational" ratios (ratos logos) to one another, while in geometry there are "irrational" ones (arratos) as well. "Theorems about sections like those in Euclid's second Book are common to both [arithmetic and geometry] except that in which the straight line is cut in extreme and mean ratio. (Heath, 1956, vol. I, p. 137)

This mention of the golden section in conjunction with analysis provides some clues, and foreshadows some of the argument to come.

Dialectic

It is difficult to give a satisfactory account of the views of both Aristotle and Plato regarding dialectic. Whereas Aristotle refers to dialectic as less than philosophy, Plato contends that it is the highest method available to philosophy. In the end, however, their methods are essentially the same, and dialectic can be viewed as having various stages. At the bottom level

dialectic is used as a means of refutation. At the top level it is a means for acquiring knowledge of real essences.

Plato openly espouses dialectic as the finest tool available in the acquisition of knowledge. "Dialectic is the coping-stone that tops our educational system" (Republic 534e).

> It is a method quite easy to indicate, but very far from easy to employ. It is indeed the instrument through which every discovery ever made in the sphere of arts and sciences has been brought to light. . . . [It] is a gift of the gods . . . and it was through Prometheus, or one like him [Pythagoras], that it reached mankind, together with a fire exceeding bright. (Philebus 16c)

Dialectic is the greatest of knowledges (Philebus 57e-58a). Through dialectic

> we must train ourselves to give and to understand a rational account of every existent thing. For the existents which have no visible embodiment, the existents which are of highest value and chief importance [Forms], are demonstrable only by reason and are not to be apprehended by any other means. (Statesman 286a)

Dialectic is the prime test of a man and is to be studied by the astronomers (Epinomis 991c). A dialectician is one who can "discern an objective unity and plurality" (Phaedrus 266b). It is the dialectician "who can take account of the essential nature of each thing" (Republic 534b). Dialectic leads one to the vision of the Good.

> When one tries to get at what each thing is in itself by the exercise of dialectic, relying on reason without any aid from the sense, and refuses to give up until one has grasped by pure thought what the good is in

> itself, one is at the summit of the intellectual realm, as the man who looked at the sun was of the visual realm. . . . And isn't this progess what we call dialectic? (Republic 532a-b)

Dialectic "sets out systematically to determine what each thing essentially is in itself" (Republic 533b). For Plato, it "is the only procedure which proceeds by the destruction of assumptions to the very first principle, so as to give itself a firm base" (Republic 533c-d). And finally, in the art of dialectic,

> the dialectician selects a soul of the right type, and in it he plants and sows his words founded on knowledge, words which can defend both themselves and him who planted them, words which instead of remaining barren contain a seed whence new words grow up in new characters, whereby the seed is vouchsafed immortality, and its possessor the fullest measure of blessedness that man can attain unto. (Phaedrus 276e-277a)

Thus, Plato indicates that dialectic is the highest tool of philosophy. Aristotle, on the other hand, appears to have a more mundane account. His passages in the Topics seem to indicate that dialectic is a tool for students involved in disputation. In fact, both the Topics and Sophistical Refutations give the impression of introductory logic texts. His references to dialectic give the appearance that dialectic does not ascend to the level of philosophy. Thus, Aristotle says,

> sophistic and dialectic turn on the same class of things as philosophy, but this differs from dialectic in the nature of the faculty required and from sophistic in

> respect of the purpose of philosophic life. Dialectic is merely critical where philosophy claims to know, and sophistic is what appears to be philosophy but is not. (*Metaphysics* 1004b22-27)

However, these considerations are somewhat misleading. Aristotle's own works seem to be much more in the line of dialectical procedure that Plato has referred to. Mayer has brought this point home forcefully through an examination of Aristotle's arguments in *Metaphysics*, Book IV. Referring to these arguments Mayer says,

> One would hardly expect the argument that Aristotle employs in the sections of Metaphysics IV. . . to be dialectical. And it is true he does not call it dialectical, but rather a kind of "negative demonstration." [i.e., at *Metaphysics* 1006a12] Yet . . . [upon examination] the arguments do appear to be dialectical. (Mayer, 1978, p. 24)

Mayer has made a very useful classification of dialectic into 3 types and their corresponding uses. I.Eristic dialectic uses confusion and equivocation to create the illusion of contradiction. II. Pedagogic dialectic is used for refutation and the practice of purification, as it leads to contradiction. III. Clarific dialectic uses criticism, revision, discovery, and clarification to dispel contradiction (Mayer, 1978, p. 1). The eristic type would be that used by a sophist. The pedagogic type is that seen in the early Socratic dialogues where the respondent is subjected to cross examination (elenchus). The third and highest type, clarific, is closer to the level of dialectic that Plato so reveres.

What Aristotle has done is

> limit the usage of the term "dialectic" to the critical [pedagogic] phase only, i.e. he sees its purpose as entirely negative, and the philosopher must go beyond this to "treat of things according to their truth". . . . By limiting "dialectic" to the negative, or pedagogic, phase of . . . dialectic, Aristotle is merely changing terminology, not method. Clarific dialectic, for Aristotle, is (or is part of) philosophical method. (Mayer, 1978, p. 1)

On this view, which I share with Mayer, Plato and Aristotle do not really differ in method. The difference is merely terminological, not substantive. This position is even more strongly supported when one considers Aristotle's response to the "eristic argument" or Meno's paradox. There Plato writes,

> A man cannot try to discover either what he knows or what he does not know. He would not seek what he knows, for since he knows it there is no need of the inquiry, nor what he does not know, for in that case he does not even know what he is to look for. (Meno 80e)

Aristotle's response is to establish a halfway house[8] between not knowing and knowing in the full sense. The crucial distinction is between the weak claim of knowing "that" something is, and the strong claim of knowing "what" something is. The former is simply to know or acknowledge the existence of a kind. But to know "what" something is, is to know the real essence of the thing. Of course this latter knowledge is the goal of dialectic according to Plato. The solution lies in the distinction Aristotle draws between nominal and real essences (or definitions). Thus, Aristotle argues that a nominal definition is a

statement of the meaning of a term, that is, meaning in the sense of empirical attributes which appear to attach to the thing referred to by the term. An example he uses of a nominal essence (or definition), is the case of thunder as "a sort of noise in the clouds" (Posterior Analytics 93b8-14). Another nominal definition is that of a lunar eclipse as a kind of privation of the moon's light.

A real definition (or essence), on the other hand, is a "formula exhibiting the cause of a thing's existence" (Posterior Analytics 93b39). For both Plato and Aristotle we ultimately must seek to know the real definition to fully know the essence of the "kind." That is to say, to have scientific knowledge of a thing one must ascend to knowledge of its real essence. However, one must begin with the former, the nominal essence. This is nothing more than to acknowledge the fact that a kind of thing exists, through an enumeration of its defining attributes. The setting forth of a nominal essence presupposes the existence of actual samples (or instances, events) in the world answering to the description contained in the definition. In other words it is not just a matter of knowing or not knowing. We begin by knowing "that" something exists. Then we proceed to seek to discover its real essence, "what" it is. The real essence of, for example, lunar eclipses (as correctly set forth by Aristotle) is the interposition of the earth between the sun and the moon. It is this "interposition" which is the

real essence which gives rise to (i.e., causes) that phenomenon we have described in our nominal definition as a lunar eclipse.

The important point here is that whether one adopts the Platonic notion of knowledge through reminiscence or the Aristotelian distinction between nominal and real essences, in both cases there is an analytic ascent to the real essence. Thus, Plato proceeds by dialectic to the real essences and first principles. Likewise, Aristotle seeks the real defining essence through an analytic ascent from things more knowable to us to things more knowable in themselves, that is, from the nominal to the real essence. Aristotle may use different terminology, but his method has its roots in the Academy. At one point Aristotle makes it clear that he sees this underlying ascent to essences and first principles in dialectic. "Dialectic is a process of criticism wherein lies the path to the principles of all inquiries"[9] (Topics 101b4). Elsewhere Aristotle makes it manifestly clear that he appreciated Plato's thought on the upward path of analysis to real essences and the first principles.

> Let us not fail to notice, however, that there is a difference between arguments from and those to the first principles. For Plato, too, was right in raising this question and asking, as he used to do, "are we on the way from or to the first principles?"[10] (Nicomachean Ethics 1095a30-35).

On the other hand, the beginning stages of dialectic occur for Plato when one attempts to tether a true belief to its

higher level cause through the giving of an account (logos). Only when one reaches the highest level of noesis is there no reflection upon sensible things.

Meno and Theaetetus

In the *Theaetetus*, Plato acts as midwife to his readers, just as, Socrates acts as midwife to Theaetetus. I believe the dialogue is consistent with the views unfolded in the *Meno* (as well as the *Phaedo* and *Republic*). In the *Meno*, Socrates indicates to Meno that true beliefs are like the statues of Daedalus, "they too, if no one ties them down, run away and escape. If tied, they stay where they are put" (*Meno* 97d). Socrates then goes on to explain how it is the tether which transmutes mere true belief into knowledge.

> If you have one of his works untethered, it is not worth much; it gives you the slip like a runaway slave. But a tethered specimen is very valuable, for they are magnificent creations. And that, I may say, has a bearing on the matter of true opinions. True opinions are a fine thing and do all sorts of good so long as they stay in their place, but they will not stay long. They run away from a man's mind; so they are not worth much until you tether them by working out the reason. That process, my dear Meno, is recollection, as we agreed earlier. Once they are tied down, they become knowledge, and are stable. That is why knowledge is something more valuable than right opinion. What distinguishes the one from the other is the tether. (*Meno* 97e-98a)

It is this providing of a tether or causal reason that assimilates a true belief of the realm of pistis, to knowledge of the realm of dianoia. The tether is arrived

at by an upward grasp of the causal objects of the next higher ontological and epistemological level. My view is that the Phaedo and Republic expand upon this upward climb using the method of hypothesis on the way. The Republic goes beyond the tentative stopping points of the Phaedo, maintaining that knowledge in the highest sense can only be attained by arriving at the unhypothetical first principle, the Good. Nevertheless, in the Phaedo, the basic method of tethering by reasoning backwards to a higher premise is maintained.

> When you had to substantiate the hypothesis itself, you would proceed in the same way, assuming whatever more ultimate hypothesis commended itself most to you, until you reached the one which was satisfactory. (Phaedo 101d-e)

This method is, of course, depicted by Socrates as being second best. The best method which continues until arrival at the Good, or One, is depicted in the Republic. The Theaetetus then is a critique of the relative level of knowledge arrived at in the state of mind of dianoia. The question is asked as to what is an adequate logos (or tether). Thus as Stenzel says,

> . . . even λόγος is included in the general skepticism. The word may indicate three things: (a) speech or vocal expression, as contrasted with the inner speech of the mind, (b) the complete description of a thing by an enumeration of its elements, (c) the definition of a thing by discovery of its distinctive nature, διαφορότης. The third of these meanings seems at first to promise a positive criterion of knowledge. But, look more closely: it is not the thing's διαφορότης, but knowledge of its διαφορότης which will constitute knowledge; and this is circular. (Stenzel, 1940, pp. xv-xvi)

This has, in general, been taken as an indication of the Theaetetus' negative ending. However, it appears to me to be another case of Plato's obstetric method. Even if one arrives at a satsifactory account or logos, and hence, arrives at a tether, it does not follow that there would be absolute certainty. "If we are ignorant of it [the Good or One] the rest of our knowledge, however perfect, can be of no benefit to us" (Republic 505a).

Thus, in the Theaetetus, Socrates indicates that, "One who holds opinions which are not true, will think falsely no matter the state of dianoias" (Theaetetus 188d). But this is a critique only relative to certainty. To arrive at tethered true opinions at the state of mind of dianoia is, nevertheless, a kind of knowledge, albeit less than the highest kind of knowledge of the Republic. On my view Haring has moved in the correct direction of interpretation when she writes:

> [the Theaetetus has] . . . a partly successful ending. The latter has to be discovered by readers. However the dialogue itself licenses and encourages active interpretation. The last ten pages of discourse contain so many specific clues that the text can be read as a single development ending in an affirmative conclusion. There is indeed a way to construe "true opinion with logos" so it applies to a cognition worthy of "episteme." (Haring, 1982, p. 510)

As stated in the previous section (supra pp. 36-38), dialectic is a movement from nominal essences to real essences. From the fact that a particular "kind" exists, one reasons backwards to the real essence, causal

explanation, or tether of that kind. From the nominal existence of surds, Theaetetus reasons backwards to the grounds or causal tether. By doing so, Theaetetus arrives at (or certainly approaches) a real definition. The relevant passage in the Theaetetus begins:

> Theaetetus: Theodorus here was proving to us something about square roots, namely, that the sides [or roots] of squares representing three square feet and five square feet are not commensurable in length with the line representing one foot, and he went on in this way, taking all the separate cases up to the root of seventeen square feet. There for some reason he stopped. The idea occurred to us, seeing that these square roots were evidently infinite in number, to try to arrive at a single collective term by which we could designate all these roots. . . . We divided number in general into two classes. Any number which is the product of a number multiplied by itself we likened to a square figure, and we called such a number "square" or "equilateral."
> Socrates: Well done!
> Theaetetus: Any intermediate number, such as three or five or any number that cannot be obtained by multiplying a number by itself, but has one factor either greater or less than the other, so that the sides containing the corresponding figure are always unequal, we likened to the oblong figure, and we called it an oblong number.
> Socrates: Excellent. And what next?
> Theaetetus: All the lines which form the four equal sides of the plane figure representing the equilateral number we defined as length, while those which form the sides of squares equal in area to the oblongs we called roots [surds] as not being commensurable with the others in length, but only in the plane areas to which their squares are equal. And there is another distinction of the same sort in the case of solids. (Theaetetus 147d-148b)

What Theaetetus has managed to do is analogous to Aristotle's discussion of lunar eclipses (supra p. 37).

Just as one reasons backwards to the real essence of lunar eclipses (i.e., the interposition of the earth between the sun and moon), so one reasons backwards to the real essence of surds. They are not "commensurable with the others in length [first dimension], but only in the plane areas [second dimension] to which their squares are equal" (Theaetetus 148b).

Theaetetus has successfully tethered the true belief of Theodorus regarding surds, by working out the reason. This is hinted at in Socrates' reply: "Nothing could be better, my young friends; I am sure there will be no prosecuting Theodorus for false witness" (Theaetetus 148b). Socrates' statement, on the one hand, implies success, and, on the other hand, suggests that Theodorus had initially stated a true belief. Thus, there is the indication of a possibly successful tethering (by Theaetetus) of a true belief (that of Theodorus).

But Theaetetus is then asked to "discover" the nature of knowledge. A very hard question indeed. But Socrates suggests that he use his definition of surds as a model.

> Forward, then, on the way you have just shown so well. Take as a model your answer about the roots. Just as you found a single character to embrace all that multitude, so now try to find a single formula that applies to the many kinds of knowledge. (Theaetetus 148d)

Notes

[1] References are to Charles S. Peirce, 1931-1958, Collected Papers, 8 volumes, edited by Charles Hartshorne, Paul Weiss, and Arthur Banks, Cambridge: Harvard University Press. All references to the Collected Papers are in the

standard form, citing only the volume number, decimal point, and paragraph number.

[2] It is interesting to note here that Kepler also became very interested in the golden section. In 1596 he wrote, "geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio [golden section]. The first we may compare to a measure of gold; the second we may have a precious jewel." Citation in Dan Pedoe's Geometry and the Liberal Arts, 1975, unpublished MS., Universiy of Minnesota, p. 215.

[3] One of the objections to abduction is that it is not really a formal logic like deduction. However, by weakening the conditions of validity and soundness it can be given a weak formalism. Furthermore, it more than makes up for lack of formalism in its creative discovery aspects.

[4] All citations to Aristotle are in the standard numbering form for each of his works. These will include the Topics, Prior Analytics, Posterior Analytics, DeAnima, Metaphysics, Nicomachean Ethics, Metaphysics, and Physics. All quotations are taken from Richard McKeon, ed., 1941, The Basic Works of Aristotle. New York: Random House.

[5] Consequences are generally considered in logic as proceeding deductively downwards. However, the essence of analysis or abduction is that it proceeds upwards (or backwards) ascending to antecedent principles.

[6] Reverse solution is the common factor above all else between Plato and Peirce.

[7] It might apply for example to the moral virtues. However, Plato most likely identifies the virtues with proportion. Hence, justice is identified as a proportionate harmony in the soul (Republic).

[8] In this way he is able to escape the horns of the "eristic" dilemma.

[9] Here Aristotle is in complete agreement with Plato.

[10] The basic movements are up from the sensible world to principles, and down from principles to particulars.

CHAPTER III

THE PYTHAGOREAN PLATO

The Quadrivium[1]

In the *Republic*, Plato describes the method by which the aspiring philosopher-statesman is to prepare himself to govern the state. Ultimately, after a series of conversions through succeeding states of awareness brought about by the apprehension of corresponding levels of subject-matter (*Republic* 513e), the philosopher must arrive at the "highest form of knowledge and its object," the Good (*Republic* 504e). As Plato points out, the Good is "the end of all endeavour" (*Republic* 505d). And furthermore, "if we are ignorant of it the rest of our knowledge, however perfect, can be of no benefit to us" (*Republic* 505a). It alone is the foundation of certainty, and is arrived at finally through the process of dialectic, what Plato calls, "the coping-stone that tops our educational system" (*Republic* 534e). "Anyone who is going to act rationally either in public or private life must have sight of it" (*Republic* 517c). "Our society will be properly regulated only if it is in the charge of a guardian who has this knowledge" (*Republic* 506a-b).

However, it is not until about the age of 50 that the philosopher is to attain this "vision" of the Good. Prior to this time, particular virtues must be both apparent and cultivated. Along with these virtues, a rigorous educational program must be undertaken. The object of this education is to assist the philosopher's mind in its ascent through the relative levels of awareness and their corresponding levels of subject-matter, being converted at each stage to the comprehension of a greater degree of clarity and reality.

Plato has spelled out the qualities that this individual must possess:

> A man must combine in his nature good memory, readiness to learn, breadth of vision, grace, and be a friend of truth, justice, courage, and self-control. . . . Grant then education and maturity to round them off, and aren't they the only people to whom you would entrust your state? (Republic 487a)

Thus, besides the inherent abilities, requisite virtues, and eventual maturity, it is the education that will prepare one for the role of philosopher-statesman. Of what is this education to consist? Leaving aside for the moment the final dialectical procedure, Plato's answer is clearly mathematics.

It is the thesis of this dissertation that Plato was more intimately involved with a mathematical doctrine throughout his career than is generally recognized by most modern commentators. This mathematical doctrine is at the

very foundation of Plato's epistemology and ontology. Through a careful analysis of his work, two very important features emerge: Plato's expert use of a method, later to be called by Charles Sanders Peirce, abduction, and his reverance for the golden section. The former is the real forerunner of the method of scientific discovery. The latter is a very important mathematical construct, the significance of which has been generally ignored by Platonic scholarship.[2] But to arrive at these points I must consider carefully Plato's famous Divided Line and the "notorious question of mathematicals" (Cherniss, 1945). From out of these I contend that the real significance of abductive inference and the golden section in Plato's philosophy will become apparent. In point of fact, I will be defending the assertion that Plato, following the Pythagorean mathematicoi,made mathematics the underlying structure of his philosophy, with the golden section being the basic modulor upon which space is given form. Further, I will be arguing that the mathematicals may be found in the middle dialogues. And, therefore, they are not merely a construction of Plato's later period, as some have contended. Additionally, the Good of the Republic and the Receptacle[3] of the Timaeus are to be identified respectively with the One and the Indefinite Dyad. The Indefinite Dyad in turn may have something to do with the golden section. Finally, I will argue that it is the logical method of abduction that lies at the center of Plato's reasoning

process. It is the very foundation of scientific discovery. Hence, I will be attempting to place Plato more firmly in the scientific tradition.

In one of the early "Socratic dialogues," the *Gorgias*, Plato plants the seed of a view that will blossom forth in the *Republic*. It is the view that mathematical study, geometry in particular, is closely allied to the establishment of a just and virtuous nature. There Socrates chides Callicles, saying:

> You are unaware that geometric equality is of great importance among gods and men alike, and you think we should practice overreaching others, for you neglect geometry. (*Gorgias* 508a)

In one of the middle dialogues, the *Republic*, an even stronger stand is taken. There mathematical study is not only allied to a virtuous and orderly life, but is the very "bridge-study" by which one may cross from the lower level of mere belief (pistis) to the higher level of reason (noesis). The bridge is understanding (dianoia), sometimes translated, mathematical reasoning, and is the intermediate level of awareness of the mathematician.

The components of this bridge-study are set out at *Republic* 524d-530e. These are the five mathematical sciences which include: (1) arithmetic, (2) plane geometry, (3) solid geometry, (4) harmonics, and (5) spherics (or astronomy). This is actually the Pythagorean quadrivium, but Plato has seen fit to subdivide geometry into "plane" and "solid," representing respectively number

in the 2nd and the 3rd dimensions. The intensive study of these mathematical disciplines is to occur between the ages of twenty and thirty. This mathematical study will prepare one for the study of dialectic, which will in turn ultimately lead to the Good. Plato explains that

> the whole study of the [mathematical] sciences we have described has the effect of leading the best element in the mind up towards the vision of the best among realities [i.e.,the Good]. (Republic 532c)

However, mathematical study is to begin well before this later intensive training.

> Arithmetic and geometry and all the other studies leading to dialectic should be introduced in childhood. (Republic 536d)

These elementary mathematical studies are to be combined with emphasis on music and gymnastic until the age of 17 or 18. Then, until the age of 20, emphasis is placed on gymnastics alone. According to Plato, this combination of intellectual and physical training will produce concord between the three parts of the soul, which will in turn help to establish justice in the nature of the individual.

> And this concord between them [i.e., rational, spirited, and appetitive] is effected . . . by a combination of intellectual and physical training, which tunes up the reason by training in rational argument and higher studies [i.e., mathematical], and tones down and soothes the element of spirit by harmony and rhythm. (Republic 441e-442a)

This training in gymnastics and music not only stabilizes the equilibrium of the soul, promoting a just nature in the

individual (and analogously a just nature in the soul of the polis), but also indicates that Plato may conceive of mathematical proportion (such as that expressed in harmonics or music) as underlying both the sensible and intelligible aspects of reality. This may be the reason why he draws music and gymnastics together in the proposed curriculum. Plato wants the individual to comprehend the proportional relationship common to both music and bodily movements. And because the soul participates in both the intelligible and sensible realms,[4] lying betwixt the two and yet binding them together into one whole, it is natural that the soul might contain this implicit knowledge. Then, upon making it explicit, the soul would benefit the most through the establishment of concord.

It is also evident in one of the last dialogues, the _Philebus_, that Plato feels the knowledge brought about by this study (i.e., of proportions in harmony and bodily movements), may have important implications in one's grasp, and possible solution, of the ancient and most difficult problem of the one and many.[5] Plato says,

> When you have grasped . . . the number and nature of the intervals formed by high pitch and low pitch in sound, and the notes that bound those intervals, and all the systems of notes that result from them [i.e., scales] . . . and when, further, you have grasped certain corresponding features that must, so we are told, be numerically determined and be called "figures" and "measures" bearing in mind all the time that this is the right way to deal with the one and many problem--only then, when you have grasped all this, have you gained real understanding. (_Philebus_ 17d-e)

I contend that this Pythagorean notion of discovering the role of number in things is one of the most crucial things that Plato wants his pupils to learn. Further, if one can discern how Plato saw the role of number in the Cosmos, and the ontological and epistemological consequences of his doctrine of the intermediate soul, then it may be possible to determine how he saw the sensible world participating in the Forms. It is most relevant that the question of participation was left an open question in the Academy (see Aristotle in Metaphysics 987b). Given the puzzle, each member was allowed to abduct his own hypothesis.

The Academy and Its Members

Let us first look at the members of the Academy and their interests. As Hackforth has pointed out, the Academy was "designed primarily as a training school for philosophic-statesmen"[6] (Hackforth, 1972, p. 7). If the Republic account is a correct indication of Plato's method of preparing these individuals, then the primary work carried out in the Academy would have been mathematical study and research. Most of the information we have concerning the associates of the Academy was copied down by Proclus from a work by Eudemus of Rhodes, entitled History of Geometry. Eudemus was a disciple of Aristotle. The authenticity of this history by Proclus, what has come to be called the "Eudemian summary," has been argued for by

Sir Thomas Heath. Heath says,

> I agree with van Pesch that there is no sufficient reason for doubting that the work of Eudemus was accesible to Proclus at first hand. For the later writers Simplicius and Eutocius refer to it in terms such as leave no room for doubt that they had it before them. (Heath, 1956, vol. 1, p. 35)

Other information has been preserved in the works of Diogenes Laertius and Simplicius. The important fragments preserved by Simplicius are also from a lost work by Eudemus, History of Astronomy. Unless otherwise indicated, the following condensed summary account is taken from the "Eudemian summary" preserved by Proclus (see Thomas, 1957, vol. 1, pp. 144-161).

Now we do know that the two greatest mathematicians of the 4th century B.C. frequented the Academy. These were Eudoxus of Cnidus and Theaetetus of Athens. Eudoxus is famous for having developed the "method of exhaustion for measuring and comparing the areas and volumes of curvalinear plane and solid surfaces" (Proclus, 1970, p.55). Essentially, he solved the Delphic problem of doubling the cube, developed a new theory of proportion (adding the sub-contrary means) which is embodied in Euclid Books V and VI, and hypothesized a theory of concentric spheres to explain the phenomenal motion of the heavenly bodies. Of this latter development Heath says:

> Notwithstanding the imperfections of the system of homocentric spheres, we cannot but recognize in it a speculative achievement which was worthy of the great reputation of Eudoxus and all the more deserving of admiration because it was the first attempt

> at a scientific explanation of the apparent irregularities of the motions of the planets. (Heath, 1913, p. 211)

To this Thomas adds the comment:

> Eudoxus believed that the motion of the sun, moon and planets could be accounted for by a combination of circular movements, a view which remained unchallenged till Kepler. (Thomas, 1957, vol. 1, p. 410, fn.b)

Eudoxus' homocentric hypothesis was set forth in direct response to a problem formulated by Plato. This is decidedly one of those instances where the role of abduction entered into the philosophy of Plato. Plato would present the problem by formulating what the puzzling phenomena were that needed explanation. This fits very neatly into the Peircean formula. The surprising fact of the wandering motion of the planets is observed. What hypothesis, if true, would make this anomalous phenomena the expected? To solve this problem requires one to reason backwards to a hypothesis adequate to explain the conclusion.

> We are told by Simplicius, on the authority of Eudemus, that Plato set astronomers the problem of finding what are the uniform and ordered movements which will "save the phenomena" of the planetary motions, and that Eudoxus was the first of the Greeks to concern himself [with this]. (Thomas, 1957, vol. 1, p. 410, fn.b)

At this point it is relevant to consider what was meant by the phrase "saving the phenomena." For this purpose I will quote extensively from a passage in Vlastos', Plato's Universe.

> The phrase "saving the phenomena" does not occur in the Platonic corpus nor yet in Aristotle's works. In Plato "save a thesis (or 'argument')" (Theaetetus 164a) or "save a tale" (Laws 645b) and in Aristotle "save a hypothesis" (de Caelo 306a30) and "preserve a thesis" (Nicomachean Ethics 1096a2) occur in contexts where "to save" is to preserve the credibility of a statement by demonstrating its consistency with apparently recalcitrant logical or empirical considerations. The phrase "saving the phenomena" must have been coined to express the same credibility-salvaging operation in a case where phenomena, not a theory or an argument, are being put on the defensive and have to be rehabilitated by a rational account which resolves the prima facie contradictions besetting their uncritical acceptance. This is a characteristically Platonic view of phenomena. For Plato the phenomenal world, symbolized by the shadow-world in the Allegory of the Cave (Republic 517b) is full of snares for the intellect. Thus, at the simplest level of reflection, Plato refers us (Republic 602c) to illusions of sense, like the stick that looks bent when partly immersed in water, or the large object that looks tiny at a distance. Thrown into turmoil by the contradictory data of sense, the soul seeks a remedy in operations like "measuring, numbering, weighing" (Republic 602d) so that it will no longer be at the mercy of the phenomenon. For Plato, then, the phenomena must be held suspect unless they can be proved innocent ("saved") by rational judgment. So it would not be surprising if the phrase "saving the phenomena"--showing that certain perceptual data are intelligible after all--had originated in the Academy. (Vlastos, 1975, pp. 111-112)

Returning then to the problem Plato set the astronomers, Eudoxus was not the only one to attempt to abduct an adequate hypothesis or solution. Speusippus, Plato, and Heraclides each developed a solution different from that of Eudoxus. Menaechmus, in essential respects,

followed the solution of Eudoxus. Callipus then made corrections on Menaechmus' version of Eudoxus' solution, which was then adopted by Aristotle. Each attempted to abduct an adequate hypothesis, or modification of a former hypothesis, which would adequately explain and predict the so-called anomalous phenomena.[7]

There is an even more telling reference to the abductive approach of Plato and Eudoxus in the "Eudemian summary." The reference is intriguing, because it refers to both abductive inference, under the rubric of analysis, and the golden section.

> [Eudoxus] multiplied the number of propositions concerning the "section" which had their origin in Plato, applying the method of analysis to them.[8] (Thomas, 1957, vol. 1, p. 153)

My contention is that this analysis is none other than what Peirce referred to as abduction. The essential feature of this method is that one reasons backwards to the causal explanation. Once one has arrived at the explanatory hypothesis, then one is able to deductively predict how the original puzzling phenomenon follows from that hypothesis. Analysis and synthesis were basic movements to and from a principle or hypothesis. Proclus was aware of these contrary, but mutually supportive movements. Thus, Morrow, in the introduction of his translation of Proclus' A Commentary of the First Book of Euclid's Elements, says,

> . . . the cosmos of mathematical propositions exhibits a double process: one is a movement

> of "progression" (prodos), or going forth from a source; the other is a process of "reversion" (anodos) back to the origin of this going forth. Thus Proclus remarks that some mathematical procedures, such as division, demonstration, and synthesis, are concerned with explication or "unfolding" the simple into its inherent complexities, whereas others, like analysis and definition, aim at coordinating and unifying these diverse factors into a new integration, by which they rejoin their original starting-point, carrying with them added content gained from their excursions into plurality and multiplicity. For Proclus the cosmos of mathematics is thus a replica of the complex structure of the whole of being, which is a progression from a unitrary, pure source into a manifold of differentiated parts and levels, and at the same time a constant reversion of the multiple derivatives back to their starting-points. Like the cosmos of being, the cosmos of mathematics is both a fundamental One and an indefinite Many. (Proclus,1970, p. xxxviii)

My view is that Proclus has correctly preserved the sense of analysis and synthesis as underlying the work of Plato. This notion of analysis appears to have been somewhat esoteric, not being clearly explicated in the writings of Plato. However, as I will later argue, Plato's notion of dialectic is closely allied to this concept. One is either reasoning backwards from conclusions to hypotheses, or forward from hypotheses to conclusions. Thus, as Aristotle noted:

> Let us not fail to notice . . . that there is a difference between arguments from and those to the first principles. For Plato, too, was right in raising this question and asking, as he used to do, "are we on the way from or to the first principles?"[9] There is a difference, as there is in a race-course between the course from the judges to the turning-point and the way back. For, while we must begin with what is known, things are

> objects of knowledge in two senses--some to us, some without qualification. Presumably, then, we must begin with things known to us. Hence any one who is to listen intelligently to lectures about what is noble and just and, generally, about the subjects of political science must have been brought up in good habits. For the fact is the starting-point, and if this is sufficiently plain to him, he will not at the start need the reason as well; and the man who has been well brought up has or can easily get starting-points. (Nicomachean Ethics 1095a30-b9)

Thus, in the movement towards first principles, it is a motion opposite in direction to that of syllogism or deduction. This is central to Plato's notion of dialectic. Hence, Plato says,

> That which the reason itself lays hold of by the power of dialectic, treats its assumptions not as absolute beginnings but literally as hypotheses, underpinnings, footings, and springboards so to speak, to enable it to rise to that which requires no assumption and is the starting point of all. (Republic 511b)

It should be noted that the notion of dialectic depicted here in the Intelligible world moves strictly from one hypothesis as a springboard to a higher or more primitive hypothesis. It does not begin from the observation of sensible particulars. However, a central theme throughout this dissertation will be that this same movement in abductive explanation takes place at every level for Plato, including the level of sensible particulars. Therefore, the abductive movement from the observation of irregular planetary motions to the explanation in terms of the underlying regularity discoverable in the Intelligible world is analogous to the

abductive movement purely within the Intelligible world. When we later examine the Cave simile (Republic 514a-521b) it will become apparent that each stage of conversion is an abductive movement, and hence a kind of dialectic, or analysis.

Now according to Plato, once a suitable explanatory hypothesis has been abducted, one can deductively descend, setting out the consequences. Of course in the Republic, where one is seeking certainty, this means first arriving at the ultimate hypothesis, the Good.

> . . . when it has grasped that principle [or hypothesis] it can again descend, by keeping to the consequences that follow from it, to a conclusion. (Republic 511b)

Thus, one then proceeds in the downward direction with synthesis (i.e., syllogism or deduction).

It is difficult to find clear statements about analysis in either Plato or Aristotle. However, Aristotle does liken deliberation to geometrical analysis in the Nicomachean Ethics.

> We deliberate not about ends but about means. For a doctor does not deliberate whether he shall heal, nor an orator whether he shall persuade, nor a statesman whether he shall produce law and order, nor does any one else deliberate about his end. They assume the end and consider how and by what means it is to be attained; and if it seems to be produced by several means they consider by which it is most easily and best produced, while if it is achieved by one only they consider how it will be achieved by this and by what means this will be achieved, till they come to the first cause, which in the order of discovery is last. For the person who deliberates seems to investigate and analyse in the way described as though he

> were analysing a geometrical construction . . . and what is last in the order of analysis seems to be first in the order of becoming. (Nichomachean Ethics 1112b12-24)

The other famous 4th century B.C. mathematician in the Academy was Theaetetus of Athens, who set down the foundations of a theory of irrationals which later found its way into Book X of Euclid's Elements. As Furley has pointed out, "Theaetetus worked on irrational numbers and classified 'irrational lines' according to different types" (Furley, 1967, p. 105). He furthered the work on the 5 regular solids, and is held to have contributed much to Book XIII of Euclid's Elements.

Speusippus, the son of Plato's sister Potone, succeeded Plato as head of the Academy,[10] and wrote a work entitled, On the Pythagorean Numbers. Unfortunately only a few fragments remain. According to Iamblichus,

> . . . [Speusippus] was always full of zeal for the teachings of the Pythagoreans, and especially for the writings of Philolaus, and he compiled a neat little book which he entitled On the Pythagorean Numbers. From the beginning up to half way he deals most elegantly with linear and polygonal numbers and with all the kinds of surfaces and solids in numbers; with the five figures which he attributes to the cosmic elements, both in respect of their similarity one to another; and with proportion and reciprocity. After this he immediately devotes the other half of the book to the decad, showing it to be the most natural and most initiative of realities, inasmuch as it is in itself (and not because we have made it so or by chance) an organizing idea of cosmic events, being a foundation stone and lying before God the Creator of the universe as a pattern complete in all respects. (Thomas, 1957, vol. 1, p. 77)

Furthermore, it is Speusippus who apparently rejected the Platonic Ideas but maintained the mathematical numbers.[11]

Xenocrates, who followed Speusippus as head of the Academy, wrote six books on astronomy. He is also credited with an immense calculation of the number of syllables that one can form out of the letters of the Greek alphabet. The number he derived is 1,002,000,000,000 (Sarton, 1970, vol. 1, p. 503, & McClain, 1978, p. 188, fn. 32). Unlike Speusippus, he retained the Ideas, but identified them with the mathematicals.

Another one of Plato's pupils was Philippus of Opus, who, according to the "Eudemian summary," was encouraged by Plato to study mathematics. It appears that he may have edited and published the Laws and possibly authored the Epinomis.[12] He wrote several mathematical treatises, the titles of which are still preserved.

Another pupil was Leodamos of Thasos, who, according to Diogenes Laertius, was taught the method of analysis by Plato. Again the intriguing mention of the method of analysis occurs. And it is important that it is Plato who purportedly taught Leodamos this method. In another place Proclus indicates that

> certain methods have been handed down. The finest is the method which by means of analysis carries the thing sought up to an acknowledged principle, a method which Plato, as they say, communicated to Leodamas, and by which the latter, too, is said to have discovered many things in geometry.[13] (Heath, 1956, vol. 1, p. 134)

In a footnote to the above statement by Proclus, Heath makes the very interesting comment,

> Proclus' words about carrying up the thing sought to "an acknowledged principle" suggests that what he had in mind was the process described at the end of Book VI of the Republic by which the dialectician (unlike the mathematician) uses hypotheses as stepping-stones up to a principle which is not hypothetical, and then is able to descend step by step verifying every one of the hypotheses by which he ascended. (Heath, 1956, vol. 1, p. 134, fn.1)

This is a very insightful remark by Heath, and I will return to this point when I consider the Divided Line in the Republic.

Menaechmus, a pupil of both Eudoxus and Plato, wrote on the methodology of mathematics. It is generally inferred from Eratosthenes that he discovered the conic sections. His brother, Dinostratus, applied Hippias' quadratrix in an attempt to square the circle.

Both Leon, the pupil of Neoclides, and Theudius of Magnesia wrote a "Book of Elements" in the Academy during Plato's time. Heath, in fact, conjectures that the elementary geometrical propositions cited by Aristotle were derived from the work of Theudius. Thus, Heath says,

> Fortunately for the historian of mathematics Aristotle was fond of mathematical illustrations; he refers to a considerable number of geometrical propositions, definitions, etc., in a way which shows that his pupils must have had at hand some textbook where they could find the things he mentions; and this textbook must have been that of Theudius. (Heath, 1956, vol. 1, p. 117)

Also living at this time was the Pythagorean, Archytas of Tarentum. An older contemporary and friend of Plato, there is little doubt that he had a major influence on Plato, though he may have never actually been in the Academy. It was Archytas who reduced the Delphic problem of doubling the cube to the problem of finding two mean proportionals. In the "Eudemian Summary" it is stated that,

> . . . [Archytas] solved the problem of finding two mean proportionals by a remarkable construction in 3 dimensions. (Thomas 1957, vol. 1, p. 285)

Thus, according to Van der Waerden, Archytas is responsible for the material in Bk. VIII of Euclid's Elements.

Cicero tells us that

> [it was during Plato's] first visit to South Italy and Sicily, at about the age of forty, that he became intimate with the famous Pythagorean statesman and mathematician Archytas. (Hackforth, 1972, p. 6)

And of course from Plato's 7th Letter (at 350b-c) we find that it was Archytas who sent Lamiscus with an embassy and 30-oared vessel to rescue Plato from the tyrant Dionysius (7th Letter 350b-c, Cairns and Hamilton, 1971, p. 1596).

As Thomas indicates in a footnote:

> For seven years [Archytas] commanded the forces of his city-state, though the law forbade anyone to hold the post normally for more than one year, and he was never defeated. He is said to have been the first to write on mechanics, and to have invented a mechanical dove which would fly. (Thomas, 1957, vol. 1, p.4, fn.a)

It is clear that Archytas was probably a major source of much of the Philolaic-Pythagorean doctrines that Plato gained access to. He was also a prime example of what a philosopher-statesman should be like. Furthermore, he is generally considered to be a reliable source of information on the early Pythagoreans. "No more trustworthy witness could be found on this generation of Pythagoreans" (Kirk and Raven, 1975, p. 314).

Many of the ideas of Archytas closely parallel those of Plato. Porphyry indicates this when he quotes a fragment of Archytas' lost book <u>On Mathematics</u>:

> The mathematicians seem to me to have arrived at true knowledge, and it is not surprising that they rightly conceive each individual thing; for having reached true knowledge about the nature of the universe as a whole, they were bound to see in its true light the nature of the parts as well. Thus they have handed down to us clear knowledge about . . . geometry, arithmetic and sphaeric, and not least, about music, for these studies appear to be sisters. (Thomas, 1957, vol. 1, p.5)

It seems obvious that the four sister sciences mentioned here refer to the Pythagorean quadrivium. But Archytas' above statement that "they rightly conceive each individual thing," should be contrasted with some of Plato's remarks regarding dialectic. Thus, Plato says, "dialectic sets out systematically to determine what each thing essentially is in itself" (<u>Republic</u> 533b). And further he says, the dialectician is one who "can take account of the essential nature of each thing" (<u>Republic</u> 534b).

There was, of course, also Aristotle, who was a member of the Academy for nineteen (perhaps twenty) years during Plato's lifetime. "From his eighteenth year to his thirty-seventh (367-348/7 B.C.) he was a member of the school of Plato at Athens" (Ross, 1967, p. ix). Aristotle considered mathematics to be one of the theoretical sciences along with metaphysics and physics (Metaphysics 1026a 18-20). But he did not devote any writing strictly to the subject itself, contending that he would leave it to others more specialized in the area. Nevertheless, his writings are interspersed with mathematical examples. And what he has to say regarding Plato's treatment of mathematics is of the utmost importance in trying to properly interpret Plato. His remarks should not be swept aside simply because one has difficulty tallying them with the Platonic dialogues.

Aristotle was obviously subjected to mathematical study in the Academy. Apparently he was not that pleased with its extreme degree of emphasis there. Thus, he indicates in a somewhat disgruntled tone, as Sorabji[14] has put it, that many of his modern cohorts in the Academy had so over-emphasized the role of mathematics that it had become not merely a propadeutic to philosophy, but the subject-matter of philosophy itself.

> Mathematics has come to be identical with philosophy for modern thinkers, though they say that it should be studied for the sake of other things. (Metaphysics 992a 32-34)

Theodorus of Cyrene should also be mentioned here. According to Iamblichus he was a Pythagorean. And according to Diogenes Laertius he was the mathematical instructor of Plato (Thomas, 1957, Vol. 1, p. 380). According to the "Eudemian summary," Theodorus "became distinguished in geometry" (Thomas, 1957, Vol. 1, p. 151). In Plato's dialogue, the Theaetetus, Theodorus appears with the young Theaetetus. There Theaetetus is subjected to the midwifery of Socrates. Theaetetus begins his account by describing the mathematical nature of his training at the hands of Theodorus. In the following passage it is interesting to note that we again find the Pythagorean quadrivium.

> Socrates: Tell me, then, you are learning some geometry from Theodorus?
> Theaetetus: Yes.
> Socrates: And astronomy and harmonics and aritmetic?
> Theaetetus: I certainly do my best to learn. (Theaetetus 145c-d)

In the "Eudemian summary," Proclus also mentions Amyclas of Heracleia, who is said to have improved the subject of geometry in general; Hermotimus of Colophon, who furthered the investigations of Eudoxus and Theaetetus; and Athenaeus of Cyzicus, who "became eminent in other branches of mathematics and especially in geometry" (Thomas, 1957, vol. 1, p. 153). But it is not clear when these three individuals appeared in the Academy, whether during Plato's lifetime, or shortly thereafter.

Archytas, Theodorus, Amyclas, Hermotimus, and Athenaeus aside, it is asserted of the others that "these men lived together in the Academy, making their inquiries in common" (Thomas, 1957, vol. 1, p. 153). However, it is not suggested, and it certainly should not be inferred, that they were all at the Academy simultaneously. It must be remembered that Plato ran the Academy for some forty years. And though Aristotle was there for nineteen of those years, several of the other individuals may have come and gone, appearing at the Academy at different times. But one thing is clear; there was an overriding emphasis in mathematical study and research. As Cherniss has correctly pointed out,

> If students were taught anything in the Academy, they would certainly be taught mathematics . . . that their minds might be trained and prepared for the dialectic; and this inference from the slight external tradition is supported by the dialogues, especially the seventh book of the *Republic*. (Cherniss, 1945, pp. 66-67)

On the Good

Further evidence in support of the mathematical nature of Plato's philosophy may be found in the accounts of his unwritten lecture, On the Good. Aristoxenus, like Eudemus, a disciple of Aristotle, writes in his *Elements of Harmony*, Bk. 2,

> Plato's arguments were of mathematics and numbers and geometry and astronomy and in the end he declared the One to be the Good. (Thomas, 1957, vol. 1, p. 389)

Thus, although the title of the lecture indicated that it was about the Good, the ultimate object of knowledge as expressed in the Republic, it nevertheless dealt with the mathematical subject-matter involved in the ascent there. And furthermore, the One was somehow identified with the ultimate object of knowledge, the Good. This tallies with what Aristotle has to say. Referring to Plato's use of two causes, the essential cause, the One, and the material cause, the Indefinite Ddyad or the Great and Small, Aristotle says,

> Further, he has assigned the cause of good and evil to the elements, one to each of the two. (Metaphysics 988a 13-15)

At another point Aristotle explains that

> the objection arises not from their ascribing goodness to the first principle as an attribute, but from their making the One a principle--and a principle in the sense of an element--and generating number from the One. (Metaphysics 1091b 1-4)

Aristotle objects elsewhere, fortunately for us in a very telling way. He says,

> . . . to say that the first principle is good is probably correct; but that this principle should be the One or . . . an element of numbers, is impossible. . . . For on this view all the elements become identical with species of good, and there is a great profusion of goods. Again, if the Forms are numbers, all the Forms are identical with species of good. (Metaphysics 1091b 19-27)

Aristotle goes on to argue that if evil is identified with the Dyad, Plato's Great and Small, it follows that

> all things partake of the bad except one--the One itself, and that numbers partake of it in

> a more undiluted form than spatial magnitudes, and that the Bad is the space in which the Good is realized. (Metaphysics 1092a 1-3)

The statement of Aristoxenus and the passages in Aristotle's Metaphysics appear to indicate that Plato held the One and Indefinite Dyad to be the principles of all entities, and furthermore gave them the attributes respectively of good and evil. It follows that if the Forms (as principles of all other entities) are derived from the One and Indefinite Dyad, then the elements from which they are derived are numerical and, hence, the Forms themselves are of a numerical nature. As the Forms are principles of all other entities, it would follow that number would be perpetuated throughout the Cosmos down into the sensible things as well.

Now Cherniss admits that

> Alexander himself says that in Aristotle's report of the lecture [On the Good], "the One" and "the great and small" were represented as the principles of number and the principles of all entities. (Cherniss, 1945, p. 28)

How then can the numerical principles of the One and Indefinite Dyad be the principles of all entities, including all sensible entities, unless the crucial feature is that numbers are in fact the essential characters of those entities? Aristotle makes this more explicit, saying,

> Since the Forms were causes of all other things, he thought their elements were the elements of all things. As matter, the great and small were principles; as essential reality, the One; for from the great and small, by participation in the One, come the

> Numbers. [And] he agreed with the Pythagoreans in saying that the One is substance and not a predicate of something else; and in saying that the Numbers are the causes of the reality of other things he agreed with them. (Metaphysics 987b 19-25)

There is an interesting parallel between what Aristotle says about the One in the Metaphysics, and what Plato says about the Good in the Republic. Aristotle says,

> . . . the Forms are the causes of the essence of all other things, and the One is the cause of the essence of the Forms. (Metaphysics 988a 9-11)

The latter part of this statement should be compared with what Plato has Socrates say about the Good.

> The Good therefore may be said to be the source not only of the intelligibility of the objects of knowledge [the Forms], but also of their being and reality; yet it is not itself that reality but is beyond it and superior to it in dignity and power. (Republic 508b)

Thus, there appears to be a definite identification of the Good with the One, and evil with the Dyad. And hence, for Plato, the two basic elements of the Cosmos are of a numerical nature. The further implication then is that the Forms are also numbers. This is in fact what Aristotle suggests,

> . . . the numbers are by him [Plato] expressly identified with the Forms themselves or principles, and are formed out of the elements. (De Anima, 404b24)

At another point Aristotle unequivocally asserts that, "those who speak of Ideas say the Ideas are numbers" (Metaphysics 1073a18-20). And in fact, not only are the Forms to be identified with numbers, but so are the

sensibles, although as numbers of a different class. This emerges in a passage in which Aristotle is discussing the Pythagoreans.

> When in one particular region they place opinion and opportunity, and, a little above or below, injustice and decision or mixture, and allege, as proof, that each of these is a number, and that there happens to be already in this place a plurality of the extended bodies composed of numbers, because these attributes of number attach to the various places--this being so, is this number, which we must suppose each of these abstractions to be, the same number which is exhibited in the material universe, or is it another than this? Plato says it is different; yet even he thinks that both these bodies and their causes are numbers, but that the intelligible numbers are causes, while the others are sensible. (*Metaphysics* 990a23-32)

Thus, numbers are not only the crucial feature of Forms, but also of sensible particulars.

Returning then to Plato's unwritten lecture, *On the Good*, Cherniss notes:

> It is said that Aristotle, Speusippus, Xenocrates, Heraclides, Hestiaeus, and other pupils attended the lecture and recorded Plato's remarks in the enigmatic fashion in which he made them (see Simplicius). Moreover, most of them apparently published their notes or transcripts of the lecture. . . . Aristotle's notes were certainly published under the title, *On the Good* [peri tagathou]. (Cherniss, 1945, p. 12)

Why then was the lecture delivered (and recorded) in this so-called "enigmatic fashion?" A clue to this may lie in the Phaedrus where the Egyptian King Thamus (Ammon) reprimands the god Theuth. The latter has claimed that his discovery of writing "provides a recipe for memory and

wisdom" (Phaedrus 274e). Thamus replies that it only leads to the "conceit of wisdom" (Phaedrus 275b).

> If men learn this [writing], it will implant forgetfulness in their souls; they will cease to exercise memory because they rely on that which is written, calling things to remembrance no longer from within themselves, but by means of external marks. What you have discovered is a recipe not for memory, but for reminder. And it is no true wisdom that you offer your disciples, but only its semblance, for by telling them of many things without teaching them you will make them seem to know much, while for the most part they know nothing, and as men filled, not with wisdom, but with the conceit of wisdom, they will be a burden to their fellows. (Phaedrus 274e-275b)

Then Plato has Socrates follow this with an analogy of writing to painting.

> The painter's products stand before us as though they were alive, but if you question them, they maintain a most majestic silence. It is the same with written words; they seem to talk to you as though they were intelligent, but if you ask them anything about what they say, from a desire to be instructed, they go on telling you just the same thing forever. And once a thing is put in writing, the composition, whatever it may be, drifts all over the place, getting into the hands not only of those who understand it, but equally those who have no business with it; it doesn't know how to address the right people, and not address the wrong. And when it is ill-treated and unfairly abused it always needs its parent to come to its help, being unable to defend or help itself. (Phaedrus 275d-e)

This tends to show a negative view on the part of Plato toward the publishing of one's doctrines. The reason being that written doctrines may be either misunderstood or abused by falling into the wrong hands. And if either of

these situations occur, the architect of the doctrine must be present to defend and rectify the situation. But too often this attendance is impossible.

This position is clearly consistent with what Plato has to say in the 7th Letter. There, referring to his most complete doctrine, Plato says,

> I certainly have composed no work in regard to it, nor shall I ever do so in the future, for there is no way of putting it in words like other studies. (7th Letter 341c)

Here, of course, Plato is not as concerned with misunderstanding or abuse, as he is with what appears to be a somewhat more mystical doctrine. As he goes on to say regarding this subject,[15]

> Acquaintance with it must come rather after a long period of attendance on instruction in the subject itself and of close compainioship, when, suddenly, like a blaze kindled by a leaping spark, it is generated in the soul and at once becomes self-sustaining. (7th Letter 341c-d)

Why then did so many of Plato's pupils copy down the lecture and, apparently, publish it? Why was it so important? I contend that it set forth, though still in an enigmatic fashion, the essentially mathematical underlying structure of Plato's philosophy. The parallel between the lecture On the Good and the Republic, especially Books 6 and 7, has already been touched upon. The Good of the Republic was identified with the One. And Aristotle goes on to contrast the account given in the Timaeus of the participant with the account in the so-called unwritten teaching. In the latter, the participant-receptacle of the

Timaeus is identified with the great and small(*Physics* 209b11-15 and 209b33-210a2). As already indicated from Alexander's account (following Aristotle) of the lecture *On the Good*, "the One and the Great and the Small were represented as the principles of number and the principles of all entities" (Cherniss, 1945, p. 28).

If further we accept the argument of G.E.L. Owen (Allen, 1965), that the *Timaeus* is to be grouped along with the *Phaedo* and *Republic* as a middle dialogue,[16] then it appears that the lecture *On the Good* may provide some mathematical keys to the interpretation of the middle dialogues. This is strictly inferential. However, I maintain that it is a most plausible inference that allows one to advocate a very consistent approach to Plato's thought.

When this is conjoined with the question of mathematicals (intermediate, seperate, immutable, and plural) and their relation to the ontological and epistemological function of the soul, a slightly revised view of Plato's middle dialogues will emerge. This, in turn, may shed some entirely different light upon the paradoxidcal problems posed in the *Parmenides*, *Theaetetus*, and *Sophist*,[17] and their possible solution.

The Pythagorean Influence

This mathematical structuring of Plato's philosophy suggests that he may have strongly adhered to, and further developed

developed, some of the doctrines of the Pythagoreans. And, if I am correct, he especially followed the matheimatikoi. It should be noted that in contrast to these matheimatikoi, Plato critiques the more exoteric Pythagorean akousmatikoi for getting caught up with the sensory aspects of harmonics.

> They look for numerical relationships in audible concords, and never get as far as formulating problems and examining which numerical relations are concordant, which not, and why. (Republic 531c)

That is to say, the akousmatikoi never rise above the more mundane features of harmonics. They do not formulate problems for themselves from which they can abduct hypotheses as solutions. These notions of setting a problem and formulating the conditions for solution (diorismos) are critical to the abductive movement, referred to by the ancients as analysis. In this regard, note especially Plato's Meno (see supra, pp. 36 & 39), where he refers to the method "by way of hypothesis" (Meno 86e-87b).

In the Metaphysics, Aristotle indicates that Plato was, in fact, a follower of the philosophy of the Pythagoreans, but also differed from them in some respects. He says,

> After the systems we have named came the philosophy of Plato, which in most respects followed [akolouthousa] these thinkers [i.e., Pythagoreans], but had peculiarities that distinguished it from the philosophy of the Italians. (Metaphysics 987a29-31)

Entirely too much emphasis has been placed upon the subsequent "differences" which were due to the Heraclitean influence of Cratylus, and not enough weight placed on the former words. Hackforth contends that the similarity between the Pythagoreans and Plato is much stronger than generally acknowledged. Of course, this is one of my contentions here as well. Referring to Aristotle's account of the relation between Platonic Forms and Pythagorean Numbers, Hackforth says:

> Despite the important divergences there noted, one of which is the transcendence of the Forms as against the Pythagorean identification of things with numbers, it seems clear that he regarded their general resemblance as more fundamental. Moreover the word akolouthousa [Metaphysics 987a30] is more naturally understood as implying conscious following of Pythagorean doctrine than mere factual resemblance. (Hackforth, 1972, p. 6)

Unfortunately, an extemely dualistic picture of Plato has been painted by those who accept the strict separation of the Intelligible and Sensible worlds in Plato's philosophy. This has resulted from too heavy of an emphasis being placed upon the Heraclitean influence of Cratylus upon Plato. This has led to a tendency by scholars to get stumped by the problems Plato sets in the dialogues, rather than solve them. If my Pythagorean hypothesis about Plato is correct, then many of the Platonic dialogue problems should be, if not actually soluble, then at least reasonably understandable.

My own view is that it was probably Philolaus who had the greatest impact upon the views of Plato. This influence may well have been channeled through Plato's friend, Archytas. It is important to recall that Plato's nephew, Speusippus, "was always full of zeal for the teachings of the Pythagoreans, and especially for the writings of Philolaus" (Thomas, 1957, vol. 1, p. 77). Along this line, it is interesting that the contents of Speusippus' book _On Pythagorean Numbers_ (see supra p. 59) holds a close resemblance to material in Plato's _Timaeus_ regarding the five cosmic elements and their harmonious relation in terms of ratio and proportion. And there is the assertion of Diogenes Laertius, presumably following Aristoxenus, that Plato copied the _Timaeus_ out of a work by Philolaus.

> . . . Philolaus of Croton, [was] a Pythagorean. It was from him that Plato, in a letter, told Dion to buy the Pythagorean books. . . . He wrote one book. Hermippus says that according to one writer the philosopher Plato went to Sicily, to the court of Dionysius, bought this book from Philolaus' relatives . . . and from it copied out the _Timaeus_. Others say that Plato acquired the books by securing fom Dionysius the release from prison of a young man who had been one of Philolaus' pupils. (Kirk and Raven, 1975, p. 308)

Though we need not assert plagiarism, it is entirely reasonable to suppose that a work of Philolaus' acted as a source book for Plato's _Timaeus_.

It is also noteworthy that Plato, in the Phaedo, refers to Philolaus. He has Socrates ask the Pythagoreans, Cebes and Simmias, whether they had not heard Philolaus, whom they had been staying with, talk about suicide.

> Why, Cebes, have you and Simmias never heard about these things while you have been with Philolaus [at Thebes]? (Phaedo 61d)

Plutarch hints that Plato in fact studied Pythagorean philosophy at Memphis with Simmias.

> Simmias appears as a speaker in Plutarch's dialogue De genio Socratis, where he says [578f] that he was a fellow-student of philosophy with Plato at Memphis--an interesting remark and conceivably true. (Hackforth, 1972, pp. 13-14)

It is probable then that the unnamed authority in Socrates' last tale (Phaedo 107d-115a) is Philolaus, especially with the reference to the dodecahedron. Thus, Socrates says to Simmias:

> The real earth, viewed from above, is supposed to look like one of these balls made of twelve pieces of skin, variegated and marked out in different colors, of which the colors which we know are only limited samples, like the paints which artists use, but there the whole earth is made up of such colors, and others far brighter and purer still. One section is marvelously beautiful purple, and another is golden. (Phaedo 110b-c)

Of course the dodecahedron reappears in the Timaeus as the foundation of the structure of the Cosmos.

> There was yet a fifth combination which God used in the delineation of the universe with figures of animals. (Timaeus 55c)

Very few of the Philolaic fragments remain. However, what fragments do remain provide a clue as to why Speusippus was so enthusiastic about his writings. It may also indicate why Speusippus' uncle, Plato, found his work so interesting, as well. Philolaus' fragment 12 appears as though it could have come straight out of the Timaeus.

> In the sphere there are five elements, those inside the sphere, fire, and water and earth and air, and what is the hull of the sphere, the fifth. (Santillana and von Dechend, 1969, p. 232)

It is difficult to adequately ascertain the thought of the early Pythagoreans, including Philolaus. They maintained an oral tradition in which their major tenets were guarded with great secrecy. Substantial fragments of a book on Pythagoreanism by Aristotle's pupil, Aristoxenus of Tarentum, preserved by Iamblichus, remain to bear this fact out.

> The strictness of their secrecy is astonishing; for in so many generations evidently nobody ever encountered any Pythagorean notes before the time of Philolaus. (Kirk and Raven, 1975, p. 221)

Furthermore, Porphyry, quoting another pupil of Aristotle, Dicaearchus of Messene, indicates the same thing.

> What he [Pythagoras] said to his associates, nobody can say for certain; for silence with them was of no ordinary kind. (Kirk and Raven, 1975, p. 221)

Thus, secrecy was the rule. He who would reveal the Pythagorean tenets on number faced punishment.

> There was apparently a rule of secrecy in the community, by which the offence of divulging Pythagorean doctrine to the uninitiated is said by later authorities to have been severely punished. (Kirk and Raven, 1975, p. 220)

Hence, Iamblichus maintained the tradition that

> the Divine Power always felt indignant with those who rendered manifest the composition of the icostagonus, viz., who delivered the method of inscribing in a sphere the dodecahedron (Blavatsky, 1972, vol. 1 , p. xxi).

This may well be why Plato was so cryptic in his discussion of the construction of the four elements and the nature of the fifth element in the Timaeus. He is discussing the formation of the tetrahedron, icosahedron, and octahedron out of the right-angled scalene triangles.

> Of the infinite forms we must again select the most beautiful, if we are to proceed in due order, and anyone who can point out a more beautiful form than ours for the construction of these bodies, shall carry off the palm, not as an enemy, but as a friend. (Timaeus 54a)

Plato may well have been concerned with not being too explicit about this Pythagorean doctrine. As he points out in the 7th Letter,

> I do not . . . think the attempt to tell mankind of these matters a good thing, except in the case of some few who are capable of discovering the truth for themselves with a little guidance. . . . There is a true doctrine, which I have often stated before, that stands in the way of the man who would dare to write even the least thing on such matters. (7th Letter 341e-342a)

Nevertheless, there is a tradition amongst the Neoplatonists that Plato was an initiate of various mystery

schools, including the Pythagorean school, and that he incurred much wrath for "revealing to the public many of the secret philosophic principles of the Mysteries" (Hall, 1928, p. 21).

> Plato was an initiate of the State Mysteries. He had intended to follow in the footsteps of Pythagoras by journeying into Asia to study with the Brahmins. But the wars of the time made such a trip impractical, so Plato turned to the Egyptians, and, according to the ancient accounts, was initiated at Sais by the priests of the Osirian rites. . . . There is a record in the British Museum that Plato received the Egyptian rites of Isis and Osiris in Egypt when he was forty-seven years old. (Hall, 1967, pp. 1&5)

Whatever truth there is in this matter, it is clear that Plato was greatly influenced by the Pythagoreans (and possibly the Egyptians). See Figure # 1, p. 122, for a projected chronological outline of Plato's life. Plato is, throughout the dialogues, obstetric with his readers, continually formulating problems and leaving hints for their solution. The reader is left to ponder these problems and, hopefully, abduct adequate solutions to them.

As stated earlier, Plato followed the Pythagoreans in maintaining that the principle elements of things, the One and Indefinite Dyad, are numbers. Likewise, as Aristotle has indicated, Plato also identified Forms and sensible particulars with numbers, though each with a different class of numbers. The Pythagorean influence of Philolaus upon Plato should become clear when considering one of the Philolaic fragments.

> And all things that can be known contain number; without this nothing could be thought or known. (Kirk and Raven, 1975, p. 310)

But what might it mean for Forms and sensible objects to be numbers? Thomas Taylor has preserved the later testimony of the Neopythagore=ns Nichomachus, Theon of Smyrna, Iamblichus, and Boetius, regarding the early Pythagorean identification of number with things. The following condensed version is adapted from Taylors's book (T.Taylor, 1983).

According to the later Pythagorean elucidations, the earliest Pythagoreans subdivided the class of odd numbers (associated with equality) into incomposite, composite, and incomposite-composite numbers. The first and incomposite numbers were seen to be the most perfect of the odd numbers, comparable to the perfections seen in sensible things. They have no divisor other than themselves and unity. Examples are 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc. The incomposite-composites are not actually a separate class, but merely the relationship between two or more composite numbers which are not divisible by the same divisor (other than unity). That is, they are composite numbers which are incommensurable (in a certain sense) with one another. For example, 25 and 27 are composite numbers which have no common factor other than unity. In approximately 230 B.C., Eratosthenes (see supra pp. 17-20), a later Pythagorean, developed his famous mathematical

sieve. It was a mechanical method by which the "subtle" incomposite numbers could be separated from the "gross," secondary composite numbers. These subtle and gross qualities were likened to the qualities in sensible things.

Likewise, the even numbers (associated with inequality) were divided into superabundant, deficient, and perfect numbers, the last of which is a geometrical mean between instances of the other two kinds. A superabundant even number is one in which the sum of its fractional parts is greater than the number itself. For example, 24 is a superabundant number: 1/2 x 24 = 12; 1/3 x 24 =8; 1/4 x 24 = 6; 1/6 x 24 =4; 1/12 x 24 = 2; 1/24 x 24 =1. The sum of these parts, 12+8+6+4+2+1 = 33, is in excess of 24.

A deficient even number is one in which the sum of its fractional parts is less than itself. For example, 14 is a deficient number: 1/2 x 14 = 7; 1/7 x 14 = 2; 1/14 x 14 = 1. The sum of these parts, 7+2+1 = 10, is less than 14.

A perfect even number is one in which the sum of its fractional parts is equal to itself. For example, 28 is a perfect number: 1/2 x 28 = 14; 1/7 x 28 = 7; 1/7 x 28 = 4; 1/14 x 28 = 2; 1/28 x 28 = 1. The sum of these parts is equal to the original number 28. These perfect numbers are geometric mediums between superabundant and deficient numbers. Any perfect number multiplied by 2 results in a superabundant number. Any perfect number divided by 2 results in a deficient number. Furthermore, perfect numbers are very rare, there being only four of them

between the numbers 1 and 10,000: 6, 28, 496, 8,128. The Pythagoreans saw a "resemblance" between this division of even numbers into perfect, superabundant, and deficient, and the virtues and vices of sensible things. Thus, Taylor records,

> Perfect numbers, therefore, are beautiful images of the virtues which are certain media between excess and defect. And evil is opposed to evil [i.e., superabundance to deficiency] but both are opposed to one good. Good, however, is never opposed to good, but to two evils at one and the same time. . . . [Perfect numbers] also resemble the virtues on another account; for they are rarely found, as being few, and they are generated in a very constant order. On the contrary, an infinite multitude of superabundant and deficient numbers may be found . . . [and] they have a great similitude to the vices, which are numerous, inordinate, and indefinite. (T.Taylor, 1983, p. 29)

Aristotle, in the Magna Moralia 1182a11, indicates that "Pythagoras first attempted to discuss goodness . . . by referring the virtues to numbers" (Kirk & Raven, 1975, p. 248). But the above recorded link between numbers and virtues appears to be limited to resemblance. Certainly Plato (and the Pythagoreans before him) had something much stronger in mind. This suggestion of an actual identification between numbers and Forms, and thereby sensible things, will become clearer as we proceed.

The Notorious Question of Mathematicals

Maintaining the Pythagorean mathematical influence of Plato's philosophy clearly in mind, we will now consider what has been termed "the notorious question of

mathematicals" (Cherniss, 1945, p. 75). In the *Republic*, Plato indicates that the trait of the philosopher is "love of any branch of learning that reveals eternal reality" (*Republic* 485a). The reason then that the mathematical sciences may be appropriate as a bridge to the Forms (Idea-Numbers) and ultimately the Good (One), is that their subject-matter may be eternal. In fact Plato says precisely this. "The objects of geometrical knowledge are eternal" (*Republic* 527b). The question then is whether these mathematical objects are distinct from the Forms, as a separate ontological class, or to be identified with the Forms.

Aristotle indicates that the mathematicals are, for Plato, a separate ontological class. Most modern commentators, however, have rejected this notion, at least that it was in the dialogues. Such diverse schools of interpretation as those of Cornford, Robinson, and Cherniss have all agreed in the rejection of a separate class of mathematicals. For example, Cornford, in reference to the intelligible section of the Divided Line in the *Republic*, states:

> Where the intelligible section is subdivided, clearly some distinction of objects is meant. I agree with critics who hold that nothing here points to a class of mathematical numbers and figures intermediate between Ideas and sensible things. (Cornford, 1965, p. 62)

Most of the attempts to find the mathematicals in the dialogues have centered around the Divided Line passage in

the Republic. Cherniss sees the whole question as simply a matter of "misunderstanding and misrepresentation" on the part of Aristotle (Cherniss, 1945, p. 25). Robinson sees it as a deduction in the Republic possible only on the assumed grounds of exact correspondence between the Cave simile and the Divided Line. This is an exact correspondence which he asserts cannot be maintained. Before dealing with the arguments of Cherniss and Robinson I will first examine what Aristotle, and then Plato himself, had to say.

Aristotle clearly sets out Plato's position on mathematicals.

> Further besides sensible things and Forms he [Plato] says there are the objects of mathematics, which occupy an intermediate position, differing from sensible things in being eternal and unchangeable, from Forms in that there are many alike, while the Form itself is in each case unique. (Metaphysics 987b14-18)

Thus, the mathematicals are eternal but partake of plurality. These mathematicals are given by Plato a definite ontological status separate from Forms and sensibles. Thus Aristotle states,

> Some do not think there is anything substantial besides sensible things, but others think there are eternal substances which are more in number and more real; e.g., Plato posited two kinds of substance--the Forms and objects of mathematics--as well as a third kind, viz. the substance of sensible bodies. (Metaphysics 1028b17-21)

Further, Aristotle makes it clear that it is these mathematicals with which the mathematical sciences are

concerned. It is "the intermediates with which they say the mathematical sciences deal" (Metaphysics 997b 1-3). Now if the mathematical sciences do in fact deal with these intermediate mathematical objects, then it simply follows that when we study the mathematical sciences, the objects of our enquiry are the mathematicals.

Aristotle again makes reference to the Platonic notion that the objects of mathematics are substances. He does this by clearly distinguishing the three views on the question of mathematicals. These are the views of Plato, Xenocrates, and Speusippus.

> Two opinions are held on this subject; it is said that the objects of mathematics--i.e., numbers and lines and the like--are substances, and again that the Ideas are substances. And since (1) some [Plato] recognize these as two different classes--the Ideas and the mathematical numbers, and (2) some [Xenocrates] recognize both as having one nature, while (3) some others [Speusippus] say that the mathematical substances are the only substances, we must consider first the objects of mathematics. (Metaphysics 1076a16-23)

It is important to note that Cherniss contends that one of the reasons Aristotle ascribes a doctrine of mathematicals and Idea-Numbers to Plato is because he (Aristotle) mistakenly confused the doctrines of Speusippus and Xenocrates with those of Plato. And yet in the above quoted passage there is a clear distinction made between the doctrines of the three individuals. And furthermore, it is apparent that Aristotle is writing here as a member of the Academy, and with reference to doctrines debated

therein. Thus, in the same paragraph, six lines later, he makes reference to "our school," and the fact that these questions are also being raised outside the Academy. Thus he says that, "most of the points have been repeatedly made even by the discussions outside of our school" (Metaphysics 1076a28-29). And this latter is in contradistinction to the previous discussion of the positions held within the Academy.

Furthermore, and to the great discredit of Cherniss' position, why would both Xenocrates and Speusippus uphold a doctrine of mathematicals in the Academy if this was entirely foreign to Plato? Certainly, Aristotle as a member of the Academy for nineteen years could not be so mistaken in attributing to his master a doctrine that Plato never held. The variance in doctrine occurred not with Plato's mathematicals, which both Xenocrates and Speusippus maintained. Rather, the differences involved the Ideas. Speusippus apparently rejected the Ideas altogether while keeping the mathematicals, and Xenocrates lowered the Ideas down to the ontological status of mathematicals, in the end identifying the two.

Cherniss, in fact, maintains that Aristotle also held a doctrine of mathematicals.

> Aristotle himself held a doctrine of mathematicals intermediate between pure Forms and sensibles, most of the Forms and all the mathematicals being immanent in the sensible objects and separable only by abstraction. (Cherniss, 1945, p. 77)

Cherniss apparently finds part of the basis for this latter assertion in a passage where Aristotle appears to be in agreement with the Pythagoreans regarding the non-separate aspect of mathematicals.

> It is evident that the objects of mathematics do not exist apart; for if they existed apart their attributes would not have been present in bodies. Now the Pythagoreans in this point are open to no objection. (Metaphysics 1090a28-31)

But if Cherniss is correct in his assertions, then the absurd conclusion that follows is that Plato's three leading pupils held a doctrine of mathematicals in one form or another, but Plato held none. This is a mistaken view. What Xenocrates, Speusippus, and Aristotle altered in attempts to overcome problems they may have perceived in Plato's doctrine, were not the mathematicals, but rather, the Ideas. Aristotle saw the universals as abstractions from sensible particulars, thereby denying separate ontologial status to the Ideas. Speusippus rejected the Ideas, while maintaining the mathematicals. And Xenocrates collapsed the Ideas and mathematicals into one. Surely Aristotle was correct when he attributed to Plato a doctrine of intermediate mathematicals.

In another passage Aristotle again clearly distinguishes the positions of Plato and Speusippus, one from the other, and from them the position of the Pythagoreans.

> [Plato] says both kinds of number exist, that which has a before and after being identical

> with the Ideas, and mathematical number being different from the Ideas and from sensible things; and others [Speusippus] say mathematical number alone exists as the first of realities, separate from sensible things. And the Pythagoreans also believe in one kind of number--the mathematical; only they say it is not separate but sensible substances are formed out of it. (*Metaphysics* 1080b11-18)

Aristotle then goes on in the same passage to distinguish the view of another unknown Platonist from that of Xenocrates.

> Another thinker says the first kind of number, that of the Forms, alone exists, and some [Xenocrates] say mathematical number is identical with this. (*Metaphysics* 1080b22-23)

The Divided Line

In the *Republic*, Plato sets forth three related similes in an attempt to explicate (metaphorically) his conception of the ascent of the mind (or soul) of the philosopher-statesman through succeeding stages of illumination, culminating with the vision of the Good. These three similes are that of the Sun (*Republic* 502d-509c), the Divided Line (509d-511e), and the Cave (514a-521b). They are actually analogies intended to perpetuate the notion of proportion, which is the underlying bond for Plato. Each simile is indicative of a process cf conversion. In the Sun and Cave, the conversion is to greater degress of light. In the Divided Line it is clearly conversion to higher levels of awareness. The analogy is between body and soul (mind), for in the end Plato links the similes, saying,

> The whole study of the sciences [i.e., mathematical sciences] we have described has the effect of leading the best element in the mind [soul] up towards the vision of the best among realities [the Good in the Divided Line], just as the body's clearest organ was led to the sight of the brightest of all things in the material and visible world [in the Sun and Cave similes]. (Republic 532c-d)

In each case the conversion is from an image to the original, or cause, of that image. It is clear from these similes that Plato intends the reader to grasp that he holds a doctrine of degrees of reality of the subject-matter apprehended. That is, at each level of conversion the soul (or mind) apprehends an increased level of reality, tethering the previous level, just as the eye apprehends an increased degree of light at each level, even though the brilliance is at first blinding. Analogously, through the use of these similes, Plato intends the reader to apprehend a doctine of degrees of clarity of mind (or soul). At each level of conversion the mind increases its degree of clarity. Thus, Plato uses the imagery of light in these similes to indicate the process of illumination taking place. Furthermore, each simile should be viewed as an extension of that which preceded it. Plato makes this point clear when in the Cave scene he has Socrates say to Glaucon, "this simile [the Cave] must be connected throughout with what preceded it [Sun and Line similes]" (Republic 517a-b).

The Divided Line is the most important of the similes with regard to the issues of mathematicals, abduction, and

proportion. There Plato begins by saying, "suppose you have a line divided into two unequal parts" (Republic 509d). What could Plato possibly mean by this? Here is the anomalous phenomenon: a line divided unequally. How do we explain or account for it? From this bare appearance, what could he possibly mean? What hypothesis could one possibly abduct that would make a line divided unequally follow as a matter of course, at least in the sense of some meaningfulness being conveyed?

Before answering this question, let us look at the remainder of his sentence for a clue.

> Suppose you have a line divided into two unequal parts, to represent the visible and intelligible orders, and then divide the two parts again in the same ratio [logos] . . . in terms of comparative clarity and obscurity. (Republic 509d)

We have already established the Pythagorean emphasis of mathematics in Plato. Up to now not much has been said about the emphasis on ratio (logos) and proportion (analogia). But as we proceed, it will become clear that these notions are critical in Plato's philosophy. Ratio is the relation of one number to another, for example 1:2. However, proportion requires a repeating ratio that involves four terms, for example, 1:2::4:8. Here the ratio of 1:2 has been repeated in 4:8. Thus, in proportion (analogia) we have a repeating ratio with four terms. Standing between the two-termed ratio and the four-termed proportion lies the three-termed mean.

> The relations among ratio, mean, and proportion can be brought out by distinguishing two kinds of mean, the arithmetic and the geometric. In an aritmetic mean, the first term is exceeded by the second by the same amount that the second is exceeded by the third. One, two, and three form an arithmetic mean. One, two, and four, however, form a geometric mean. In a geometric mean, the first term stands to the second in the same ratio that the second stands to the third. Either mean may be broken down into two ratios, namely, that of the first and second terms and that of the second and third. But the geometric mean alone is defined by ratio, being the case in which the two ratios are the same; all other means have two different ratios. Of all the means, therfore, only the geometric can be expanded into a proportion, by repeating the middle term. Furthermore all proportions in which the middle terms are the same . . . can be reduced to geomtric means, by taking out one of the two identical terms. (Des Jardins, 1976, p. 495)

An example of this relation of a geometric mean to proportion will help to illustrate this. The ratio 1:2 and the ratio 2:4 can be related so that we have a proportion. Thus, 1:2::2:4 is our proportion repeating the same ratio. However, because the middle terms are identical, each being 2, one of the middle 2's can be dropped to establish a geometric mean: 1:2:4. Here 2 is the geometric mean between 1 and 4. Whenever a case like this arises in which a proportion contains two identical middle terms, we can say we have the peculiar instance of a proportion which contains only three "different" terms, even though it is true that one of these terms (i.e., the middle) gets repeated. We shall nevertheless refer to such a creature as a three-termed geometric proportion.

Now given that we apprehend how three different terms can make up a proportion, a geometric proportion, is it possible to somehow arrive at such a proportion with only two terms? Let us again refer to the Divided Line. Here we have a line divided into two unequal parts. Hence it would be reasonable to say that we have two different terms here in the sense that we have two different lengths of line. For example, it might be that we have one segment being 2 units in length and another segment being 4 units in length. But of course we do not know the number of units of each segment. All we are told initially is that we have two unequal segments of line.

Let us inquire before going any further, as to why we are dealing with a line in the first place. Wouldn't it have been just as simple to deal strictly in terms of number, without magnitude entering into the discussion? Did not the Pythagoreans, whom Plato (as I have argued) relied so heavily upon, find number (i.e., arithmetic) to be the mother of the other mathematical sciences? As such, wouldn't it be easier to work just with primitive numbers rather than number extended in one dimension? The answer lies in our first abduction. Why a line? Because lines alone can be used to represent incommensurable magnitudes. Des Jardins has also recognized this fact.

> Lines as such are neither commensurable nor incommensurable, but only the incommensurable magnitudes have to be represented by lines, for by definition the commensurables can also

> be represented by numbers. It is only if the ratio of kinds indicates an incommensurable division that one needs lines instead of numbers. (Des Jardins, 1976, p. 485)

This then may be our tentative conjecture, or abduction.[18] It need not be true, but it is a good working hypothesis. Let us merely keep it in the back of our minds while we pursue the former question I posed. Is there a way in which out of the two line segments we can arrive at not merely some ratio, but a geometric proportion of three different terms as defined above? The only way to achieve this would be to consider the whole line as one magnitude, in relation to the greater segment, and the greater segment in relation to the lesser segment. Thus we could say as the lesser is to the greater, so the greater is to the whole. If we used the magnitudes originally suggested, 2 and 4, we would have 2:4::4:6. But this is a false assertion of a geometric proportion. If we had 2:4::4:8, we would have a geometric proportion, because the two middle terms are identical. However, the two segments of 2 and 4 when added together do not equal 8, but rather 6. The sad fact is that no matter what commensurable whole numbers we attach to the Line, the proportion will not come out correct. Although, in a later section (ifra p.149 ff) we will see how the terms of the Fibonacci series approximate it.

What do we do? We return to our original abduction, i.e., that Plato has chosen a line because lines are

necessary to represent incommensurable magnitudes. Now the answer should spring into our minds. There is absolutely one and only one way to divide a line, such that the lesser is to the greater in the same ratio as the greater is to the whole. This is the division in mean and extreme ratio. This is none other than that which has come to be called the golden cut or golden section. The full abduction then is: given the mere fact that we are presented with a line divided, that is, a line divided in a particular ratio which will then be cut twice again by Glaucon to perpetuate the porportion, the hypothesis that the original cut is the golden cut, is the most reasonable one to set forth.

Furthermore, the fact that Plato is essentially a Pythagorean interested in the ratios, proportions, and harmonies of the Cosmos, makes it ludicrous to suggest that he intended that the line merely have an initial indiscriminate division. As I have taken pains to show, we are dealing in the Academy with some of the most refined mathematician-philosophers of all time. We add to this the fact that Plato is obstetric in all that he does. He may not have clearly set out his deepest doctrines, but they are there for those who have eyes to see. His method was to formulate the problem, the object of his pupils was then to reason backwards to, or abduct, a hypothesis that would make the anomalous feature (set out in the problem) follow as a matter of course. I will consider this matter of the

golden section more extensively later, but for now let us return to the Divided Line.

Referring to Figure # 2(p. 123), the initial division at C separates the Intelligible sphere of knowing (episteme) BC, from the sensible or Visible sphere of opinion (doxa) AC. The lower line (doxa) is then itself divided into belief (pistis) DC and conjecture (eikasia) AD. Furthermore, this last division is expressly distinguished by different objects. These objects are then purported to be in the

> relation of image [AD] to origininal [CD] . . . the same as that of the realm of opinion [AC] to that of knowledge [BC]. (Republic 510A)

Thus, just as the reality of the visible world is of a lesser degree than, and stands in relation to, the greater reality of the intelligible world, in like manner the reality of the subject-matter of eikasia stands to the subject-matter of pistis. By analogy then, dianoia will stand in a similar relation to noesis.

Plato then goes on to distinguish the subsections of the upper line, not expressly in terms of subject-matter, but only inferentially. Rather, the distinction is made as to how the mind operates in the two subsections, though it nevertheless is true that he infers a separate set of objects for CE, the mathematicals, distinct from those of BE. The gist of the matter is that in the state of dianoia, the "students of geometry and calculation and the

like," use the objects of pistis such as visible diagrams as images (images of what is not clearly stated), and using various assumptions proceed downwards "through a series of consistent steps to the conclusion which they set out to find" (Republic 510d).

However, in the state of noesis (BE), the assumptions are treated

> not as principles, but as assumptions in the true sense, that is, as starting points and steps in the ascent to something which involves no assumption and is the first principle of everything. (Republic 511b)

Plato has Socrates then go on to say,

> . . . the whole procedure involves nothing in the sensible world; but moves solely through forms to forms and finishes with forms. (Republic 511b-c)

Thus, the objects of noesis are clearly Forms.

Although Plato does not specifically assert the existence of a separate class of objects for dianoia, he does have Socrates say two important things. First, the geometers who operate at the intermediate level of awareness

> make use of and argue about visible figures [objects of CD], though they are not really thinking about them but the originals [objects of CE]which they resemble. . . . The real objects of their investigation being invisible except to the eye of reason [dianoia]. (Republic 510d)

Secondly,

> . . . you may arrange them [the four states of mind] in a scale, and assume that they have degrees of clarity corresponding to the degree of truth possessed by their subject-matter. (Republic 511e)

The big question is whether the subject-matter of dianoia and noesis are different, or whether they are the same, i.e., Forms, but simply "seen" in light of differing degrees of intensity at the two levels of awareness. When the analogy (proportion) between the four levels is compounded with the assumption of correspondence between Line and Cave, it is reasonable to abduct the hypothesis of a separate set of objects (presumably the mathematicals) corresponding to the intermediate level of awareness, dianoia. Robinson has most succinctly summarized the reasoning as follows:

> [in the Cave simile] the unreleased prisoners regard the shadows and echoes as originals. They take them, not as means of knowing a reality beyond them, but as themselves the only reality to be known. It follows that 'conjecture' [eikasia] is not trying to apprehend originals through their images but taking the image for the original. Adding this result to Plato's statement that thought [dianoia] is to intelligence [noesis] as conjecture [eikasia] is to conviction [pistis], we obtain the probable conclusion that thought or dianoia is also a form of taking the image for the original. Since the original is in this case the Idea (for the object of intelligence or noesis is said to be Ideas), we infer that thought [dianoia] takes images of the Ideas for the realities themselves. But these images cannot be what Plato usually points to as images of the Ideas, namely the world of Becoming, for that is clearly the object of "conviction." What can they be? And how tantalizing of Plato not to say! (Robinson, 1953, p. 181)

If we follow the prior evidence of Plato's emphasis upon mathematics within the Academy and its crucial role as a bridge-study for the philosopher-statesman in the

Republic, and further, consider Aristotle's assertion that Plato held a doctrine of intermediate mathematicals which are like the Forms in being eternal; and then, combine this with Plato's remarks that the state of awareness of dianoia corresponds to that of the mathematician, it would appear that this set of objects corresponding to dianoia would be the mathematicals. This would be the obvious abductive conclusion, though one might ask the question as to why Plato does not explicitly state it as such within the dialogues, especially the Republic. For there he simply says,

> The relation of the realities correspoding to knowledge [episteme] and opinion [doxa] and the twofold divisions into which they fall we had better omit if we're not to involve ourselves in an argument even longer than we've already had. (Republic 534a)

Why the mystery? But may it not be the case that this is at the center of the oral tradition within the Academy, a tradition that has its roots in Plato's Pythagoreanism? Again, taking into consideration Plato's statements in the Phaedrus and 7th Letter regarding written doctrine, along with the one place where Aristotle clearly expresses a difference in doctrine between the dialogues and Plato's oral presentation over the question of participation (a difference based upon mathematical principles), may we not very plausibly infer that Plato's doctrine of mathematicals is a central doctrine not to be explicitly, though inferentially, exposed within the dialogues? In fact, even

in the later period dialogues, the mathematicals are not explicitly exposed.

If Plato is acting as a midwife in the dialogues, as I contend he is, then he has planted sufficient seeds for the reader to reasonably abduct a hypothesis of intermediate mathematicals. Robinson mistakenly maintains that the mathematicals must be deduced from an exact correspondence between Line and Cave. And further, he argues that if the exactness of the correspondence breaks down, then any supposed deduction of mathematicals is invalid (Robinson, 1953). But the requirement of deduction is too onorous. The conjectural notion of abduction is much more appropriate, and is consistent with Plato's underlying method. Furthermore, deduction or synthesis follows only upon the result of arriving at the hypothesis through abduction or analysis. Hence, "exact" correspondence between Line and Cave is not necessary. However, the question of correspondence of Line and Cave is relevant, and therefore, it is appropriate to consider Robinson's argument.

Robinson's argument is essentially that if there were an "exact" correspondence between Line and Cave, then the state of the unreleased prisoners in the Cave would be equivalent to the initial stage of the Line, eikasia, and the indiviual would at that stage be taking images for originals. And furthermore, the move from eikasia to

pistis would be analogous to the initial release of the prisoners, turning from the shadows to the objects which have cast them. If this were not the case, Robinson argues, then it would break the back of the correspondence argument from which the mathematicals may be "deduced" (Robinson, 1953, p. 183). Regarding this latter point, Robinson argues as follows:

> There is one point at which the correlation must break down completely whatever interpretation we assume of the debated questions; and Professor Ferguson has shown very forcibly what it is [see Ferguson, 1934, p. 203]. If there were a precise correlation, the state of the unreleased prisoner would have to be "conjecture" (eikasia), and the state immediately succeeding his release would have to be the adjacent state in the Line, namely pistis. But pistis which means conviction or confidence and refers at least primarily to our ordinary attitude to "the animals about us and all that grows and everything that is made" [Republic 510a], bears no resemblance to the prisoner's condition immediately after his release; for the latter is expressly described as bewilderment and as the belief that his present objects are less real than his previous objects [Rebublic 515d]. In view of this observation we must say that Plato's Cave is not parallel to his Line, even if he himself asserts that it is. (Robinson, 1953, p. 183)

Is Robinson's contention true? To answer this let us first look at the state of eikasia. Plato intends that we understand by eikasia (AD), in addition to mere perceptual illusions, a purposefully distorted set of objects or notions which create a false conjecture, or illusion in the mind of the recipient. It is the realm of false opinion in which the copies are distortions of the facts (originals)

of the realm of pistis (CD). This level is one of corruption. It is, in fact, like the bound prisoners of the Cave simile, who see only shadows and never the causes of those shadows (Republic 514a). It is the state of mind of those who have been corruped by sophists, rhetoricians, politicans, imitative artists, false poets and dramatists. It is the realm of those involved in "their shadow battles and struggles for political power" (Republic 520c). Here imitative art deludes the individual, as Plato points out in Book 10 of the Republic. It is an "inferior child [level AD] born of inferior parents [level CD] (Republic 602c).

Hamlyn links eikasia with the state of the bound prisoners of the Cave scene, and further, identifies the move from eikasia to pistis with the freeing of the prisoners from chains as a result of Socratic enquiry [elenchus] (Hamlyn, 1958, p. 22). However, I would add to this that the initial move, as well as containing the negative Socratic element, also contains a positive Platonic element, and is not completed until the individual apprehends a new object of reference, an object corresponding to pistis (CD).

Hamlyn points out that AD is a caricature of the philosophical views of a Protagoras or a Hume.

> The position of the prisoners in the Cave [bound to AD] is not that of the ordinary man but of a sophist like, Protagoras, as Plato sees him . . . [or] those who have been corrupted by the sophist. . . . this view makes more plausible the view that the first

> stage in the ascent towards illumination is being freed from chains. It is not chains that bind the ordinary man unless he has been corrupted by others; the most that can be said of him is that he is lacking sophistication. The trouble with the sophists and their disciples is, according to Plato, that they do possess sophistication, but sophistication of the wrong sort. Consequently the first stage in their treatment is freeing them from their false beliefs so that they can return to the ordinary unsophistication which can itself be a prelude to the acquisiton of "true sophistication." The first stage is ordinary Socratic practice. (Hamlyn, 1958, p. 22)

These false beliefs do have their objects of reference. They are distortions of facts, like their originals in that they pertain to factual material, but unlike them in that they are purposefully distorted. The individual bound to the level of awareness of eikasia accepts these false beliefs as though they were true beliefs based on real facts. Thus, distortions or illusory images of the originals are accepted in place of the originals. Hence, contrary to Robinson's first point (see supra), the state of eikasia is to be identified with the state of the unreleased prisoners in the Cave. In both cases the individual is accepting images for the originals. The acceptance of these false beliefs is overturned by breaking the hold that the illusory images have on the mind of the individual. This is done through the Socratic elenchus. Entailments of one's belief are drawn into question, as are the supposed factual underpinnings upon which it is based. The more positive Platonic aspect is begun when a new set of objects of reference, i.e., the

originals, are displayed to the individual. It is the apprehension of these new objects of reference (and all of their logical relations) which brings about a new state of awareness, pistis. This is depicted in the Cave simile when the formerly bound prisoner turns toward the objects which cast the shadows. Though at first "he would be too dazzled to see properly the objects of which he used to see only the shadows" (Republic 515d).

Robinson's second point (following Ferguson) is that the initial move in the Line from eikasia to pistis cannot lead to bewilderment as it does in the Cave. But is this true? Is it not the case that at each level of conversion there is an initial state of bewilderment and confusion, both before the eyes (in the case of the visible realm) or the mind's eye (in the case of the intelligible realm) can be adjusted to the new level of reality? Take as a purely hypothetical example the case in which an individual has been corrupted by the influence of sophistical politicians. These politicians have actually instilled in the individual's mind (through falsified evidence) the false belief that the assassination of a former President of the United States was the personal effort of an extreme left-wing advocate of communist policy with Russian anti-capitalistic connections. Further, our subject has been persuaded that the assassin acted alone, shooting the President with a single shot. And further assume for the moment that other corrupt individuals (politicians and

judges) falsified and distorted evidence which enhanced the conviction in the mind of the individual that these were actual facts and that his corresponding opinions, doxa, were unquestionably true. If our subject is then presented with the "real facts" that indicate that the assassin was actually a member of a group of assassins who were solicited by high government and military officials, it will not lead simply to a change of opinion. Further, assume that sufficient data are revealed in terms of government documents, tape recordings, films, public testimonies and admissions. Is not the individual going to be pressed into an initial state of bewilderment? Surely, if we may assume for the moment that sufficient tangible facts are revealed and the truth of the matter is unfolded, the individual will abandon his false conjecture and take on a new belief. But it is not a mere change of opinion; rather, it is a completely new situation. In the first instance, the individual was purposefully given falsified, illusory data (objects of eikasia), actual distortions of the real facts. And, correspondingly, his mind held a distorted and illusory conjecture. In the second instance, entailments and underpinnings of the false conjecture were drawn into question by a new set of "real facts" or evidence. This, in turn, led to the rejection of the false hypothesis which the individual had been led to accept by the sophistical influence of the corrupt politicians. This results in an initial state of perplexity, as prior

convictions are broken down with an associated destruction of faith in politicians and certain governmental institutions. The perplexity precedes the actual acceptance of a new hypothesis (or belief) based on the new set of "real facts," i.e., the originals of pistis. The important point is that any move from eikasia to pistis involves this initial state of bewilderment , contrary to the aforementioned contention of Robinson (Robinson, 1953) following Ferguson.

This initial state of confusion is, in fact, evident at each level of conversion, whether one is considering the perception of the visible realm through the eyes or the apprehension of intelligibles by the mind's eye. It appears that both Robinson and Ferguson have ignored a very relevant passage where Plato explicitly makes this point:

> Anyone with any sense will remember that the eyes may be unsighted in two ways, by a transition either from light to darkness or from darkness to light, and will recognize that the same thing applies to the mind. So when he sees a mind confused and unable to see clearly he will not laugh without thinking, but will ask himself whether it has come from a clearer world and is confused by the unaccustomed darkness, or whether it is dazzled by the stronger light of the clearer world to which it has escaped from its previous ignorance. (Republic 518a-b)

Thus, Robinson is incorrect in both of his assertions. But he further argues that it will be impossible to discern in the Cave the four states found in the Line. But is this notion of "exact correspondence" necessary? If I am

correct that Plato is concerned with ratio and proportion, then we should examine the relation of Sun to Line to see if there is a ratio which is then perpetuated in terms of the relation between Line and Cave. Referring to Figure #2 we notice that the Sun simile contains an initial twofold division into the Intelligible and Visible realms. In the Line, each of these realms is then further bifurcated. This gives us a fourfold division into noesis, dianoia, pistis, and eikasia. This gives the ratio 2:4. If this ratio is to be extended into the Cave scene to create a geometric proportion then we would expect there to be an eightfold division there. This would give the geometric proportion of 2:4::4:8. However, because the middle term repeats itself, we can collapse the formula to 2:4:8. Hence, this would mean that the ratio of 2:4 is perpetuated through the similes. But 2:4 is really reducible to 1:2. or the relation of the One to the Dyad.

Looking at the Cave scene, there does appear to be a plausible eightfold division. The first stage would be that of the shackled prisoners who view only the shadows projected onto the wall, and take them to be the real (Republic 514b-515c). The second stage would occur when the prisoner is unshackled, and turning, views the objects themselves which had cast the shadows (515c-d). The third stage would occur when he looks directly at the light of the fire (515e). The fourth stage would occur when he is forcibly dragged out of the cave into the sunlight.

Although

> his eyes would be so dazzled by the glare of it that he wouldn't be able to see a single one of the things he was now told were real (515e-516a).

The fifth stage would occur when he views the shadows (516a). The sixth stage would occur when he looks "at the reflections of men and other objects in water (516a). The seventh stage would occur when he looks at the objects themselves (516a). And the eighth stage would occur when he looks at the planets and stars in the light of the moon at night (516a-b).

One might legitimately ask at this point as to why there is no stage corresponding to the vision of the sun itself? The answer lies in the fact that if we return to the Line, we notice that even though there is a final vision of the Good, it does not act so as to make a fifth division of the Line. Rather, the Good (or One) sits at the summit of noesis. Analogously, the sun in the Cave scene sits at the summit of the upward ascent. And conformably to the nature of the Good in the Line, there is no extra stage set out in the Cave. Just as the Good is the last upward move in the Line and the foundation for certainty (505 a&d), so the vision of the sun in the Cave scene, "must come last" (516b). Hence, on this view we have the ratio of the One to the Dyad, 1:2, perpetuated through the three similes in the geometric proportion 2:4:8.

Robinson, however, goes on to assert that the changes in the Cave do not

> fit precisely . . . the states enumerated in the Line. Plato's intention seems rather to describe a single continuous change terminated in both directions. (Robinson, 1953, p. 182)

To this he adds the remark that the change is "infinitely divisible within those bounds" (Robinson, 1953, p. 182). The notion that the change is terminated in both directions is correct. However, the whole process of conversion must be conceived of as a finite number of discrete steps ending with the contemplation of that which sits at the summit. This is supported by the evidence that Plato abhorred the concept of an infinitely divisible continuum. His response was to postulate the existence of indivisible lines. Otherwise one would be continually going through conversions without ever reaching the summit, if one could even get started in the first place, much as in Zeno's paradox of the stadium (see Physics 239b11-12 & Topics 160b7-8). Thus, Aristotle says,

> Further, from what principle will the presence of the points in the line be derived? Plato even used to object to this class of things as being a geometrical fiction. He gave the name of principle of the line--and this he often posited--to indivisible lines. (Metaphysics 992a19-22)

And as Furley has argued,

> Aristotle did attribute to Plato a theory of indivisible lines, and he did understand this to mean that there is a lower limit to the divisibility of extended magnitude. (Furley, 1967, p. 107)

What does Plato himself say about the correspondence between Sun, Line, and Cave. Plato has Socrates say to

> This simile [the Cave] must be connected throughout with what preceded it. The realm revealed by sight corresponds to the prison, and the light of the fire in the prison to the power of the sun. And you won't go wrong if you connect the ascent into the upper world and the sight of the objects there with the upward progress of the mind into the intelligible region. (Republic 517b-c)

This is clearly an unequivocal connection of Cave and Line, placing them in correspondence.

Now, there is a very big difference between denying "exact correspondence" of Line and Cave, on the one hand, and straightforward correspondence of Line and Cave, on the other. Our first effort is surely to understand the intentions of Plato, not to construct a theory and then forcibly fit him into it. Sometimes commentators will go so far as to literally deny Plato's very words. Thus, Robinson says, "we must say that Plato's Cave is not parallel to his Line, even if he himself asserts that it is" (Robinson, 1953, p. 183)! At other times they will reject the words of Plato's most prolific disciple, Aristotle, as being mere misunderstanding and misrepresentation. This was the case with Cherniss, as cited earlier (see supra., p. 85).

Plato's Cave and Line may not be in exact correspondence, but they are in correspondence, as Plato intends them to be, and in fact, states that they are. This is true irrespective of whether or not my eightfold division of the Cave is correct. This correspondence

provides sufficient grounds upon which to base the abductive inference to the hypothesis of mathematicals. In the first place both Cave and Line are terminated in both directions (up and down). In both similes the starting-point and ending-point are the same. That is to say, the state of the bound prisoner in the Cave taking the shadows as the real objects is the analogue of the state of eikasia in the Line in which the individual holds false conjectures having accepted as real the distorted images of supposed facts presented him by corrupt politicians, sophists, rhetoricians, etc. And the culminating vision of the sun in the Cave scene is the analogue of the final vision of the Good in the Line. In the second place, in both Line and Cave the ascent is made by a series of conversions from what is in each case a copy of less reality (that has been accepted as the real) to an original of greater reality. In particular we have seen that the conversion from eikasia to pistis may be identified with the release and turning around of the prisoner to face a new set of objects of reference, the originals of the formerly accepted distorted images.

The important point is that there is adequate correspondence between Line and Cave to assert that each level of awareness accepts a copy of a higher reality with the conviction that it is the real, right on up until we arrive at the truly real, the unhypothetical first

principle, the Good. This results primarily from the identification of eikasia with the unreleased prisoners watching the shadows, on the one hand, and the geometrical proportion existing between the sections of the Divided Line.

The argument may be stated in the following way. Where L is eikasia, M is the state of the bound prisoners, N is taking a copy for the original:

1. L=M,
2. M has property N,
3. Therefore, L has property N.

Then assume pistis is X, dianoia is Y, and noesis is Z. Taking a copy as "the real" is relative to the original of which it is a copy. Thus, eikasia takes its object of reference relative to the original in pistis. The proportion involved is L:X::Y:Z. If L has the property N in relation to X, then Y has the property N in relation to Z. Eikasia does in fact have this property N in relation to pistis, hence, it follows that dianoia has the property N in relation to noesis. What the argument leads to is the inference that there is a set of objects of reference for dianoia which are copies of the Forms of noesis, and which thereby are distinct ontologically from the Forms in that they partake of a lesser degree of reality. Prior to the conversion to the state of noesis, these copies are accepted as the real. What objects occupy this space? They must be eternal like the Forms because they also exist in the Intelligible region. There must be some other

criterion of distinction. If we now consider that the state of awareness of dianoia is identified with that of the mathematician, then it is reasonable to infer that the objects we are looking for would be those with which the mathematician is concerned. These are not the diagrams which he draws (the objects of pistis). These diagrams are mere images of the intelligible mathematical objects upon which the mathematician is now focussed. Thus, dianoia is most probably concerned with non-sensible mathematial objects. What the argument leads to is the need for a set of objects which are intelligible (hence eternal), yet which possess less reality than their originals the Forms.

If I am correct in asserting that mathematics provides the scaffolding for Plato's philosophy, and further that the Forms are numbers and mathematical concepts (i.e., numbers, lines, planes, solids, equality, etc.), and if the objects of dianoia are images of these Idea-Numbers, then they will themselves be mathematical objects or concepts.

The question then is whether the mathematicals attributed to Plato by Aristotle fulfill the requirements of this class of objects corresponding to dianoia. Aristotle's assertions seem to indicate that they do.

> Further besides sensible things and Forms he says there are the objects of mathematics, which occupy an intermediate position, differing from sensible things in being eternal and unchangeable, from Forms in that there are many alike, while the Form itself is in each case unique. (<u>Metaphysics</u> 987b14-18)

Thus, both Forms and mathematicals being eternal, the latter differ from the former in their plurality. On the hypothesis that the Forms are Idea-Numbers, there is no difficulty in conceiving of the mathematicals as images of the Forms. The mathematicals simply participate to a greater degree in plurality, possibly in the Indefinite Dyad. It is their plurality which distinguishes the mathematicals from the Forms. They clearly fill the role left open by the inference of a less real class of objects which are images of the Forms.

Furthermore, Plato clearly indicates that "the objects of geometrical knowledge are eternal" (Republic 527b). Geometrical knowledge is a subclass of mathematical knowledge. Plato says that the state of awareness of a mathematician is dianoia. Thus, the mathematician operating at the level of dianoia has eternal objects as the referents of his mathematical knowledge. Now Plato says that "different faculties have different natural fields" (Republic 487a-b). But the objects of dianoia are eternal, though they can't be the Forms, for the reasons just suggested. Again the argument leads to the abductive suggestion of a separate class of eternal objects, differing from the Forms. Furthermore, it follows from the fact that the state of dianoia is the state of awareness of the mathematician, that the appropriate objects of reference will be mathematical. Thus, there appears to be the need for a class of eternal mathematical objects,

distinct from the Forms. Aristotle attributes this ontologically intermediate class of objects to Plato, and it appears to satisfy all of the requirements brought out in the Republic.

Alternatively one can arrive at the need for an ontologically separate class of mathematicals without appealing to any correspondence between Line and Cave at all. The derivation is contained within the Line itself in relation to Plato's statement about the lover of knowledge.

> Our true lover of knowledge naturally strives for reality, and will not rest content with each set of particulars which opinion takes for reality. (Republic 490b)

Thus, doxa, in general, tends to take its objects (images relative to the objects of episteme in general) as reality. One of the primary purposes of the Divided Line is to transmit the basic features of the relation between Itelligible and Visible worlds (and of course man's epistemological relation to those worlds) into the subsections of this major division. Thus, just as man tends to take the sensible particulars (images of the objects of the Intelligible region) as real, so within the subsections he at different stages takes the objects in eikasia (images of the objects of pistis) for the real, and analagously takes the objects of dianoia (images of the Forms) for the real. It must be remembered that in each set of circumstances what is "the real" is relative to the level of reality of the prior and posterior ontological and

epistemological levels. Employing the notation used in the Divided Line Figure #2 (p. 123), we have, AC:BC::AD:CD::CE:BE. That is to say, as doxa is to episteme, so eikasia is to pistis, and so dianoia is to noesis. Thus, just as doxa is taking an image (relative to the original of episteme) for the real, so dianoia is taking an image (relative to the original of noesis) for the real.

Plato employs other metaphors to set forth the analogous positions of eikasia and dianoia, thereby indicating that in both cases one is taking an image of the original for the real. Thus, as regards dianoia he says,

> . . . as for the rest, geometry and the like, though they have some hold on reality, we can see that they are only dreaming about it. (Republic 533b-c)

Analogously, those tied to the world of conjecture, more specifically to the sophistical contentions involved in the state of eikasia, are also dreaming. Only when the philosopher-king descends from his vision of the Good will the spell of eikasia be broken, and the polis awaken from its slumber and not be "merely dreaming like most societies today, with their shadow battles and their struggles for political power" (Republic 520c).

If my argument is correct, there is sufficient material within the Republic to abduct the hypothesis of an ontologically intermediate level of mathematical objects. But there is really no clear statement regarding them.

There is one passage though that is very suggestive, where Socrates says to Glaucon, if one were to ask the arithmeticians

> "this is very extraordinary--what are these numbers you are arguing about whose constituent units are, so you claim, all precisely equal to each other, and at the same time not divisible into parts?" What do you think their answer would be to that? [Glaucon to Socrates:] I suppose they would say that the numbers they mean can be apprehended by reason [dianoia], but that there is no other way of handling them. (Republic 526a)

In a footnote to Repbulic 526a, Lee makes the following remarks,

> The language of the previous paragraph [Republic 524d-e] ("number themselves," "the unit itself") is that of the theory of Forms. It is less clear what are the numbers referred to in this sentence. Some have supposed them to be entities intermediate between forms and particulars. . . . But though Plato did hold some such view later in his life, this sentence is very slender evidence for them in the Republic. (Lee, 1980, p. 333)

This is not an atypical reaction. But where does one distinguish between the earlier and later view as regards the mathematicals. Aristotle makes no such distinction in his writings.[19] However, he dose provide a further clue as to what these numbers made up of precisely equal units and yet indivisible may be. In contrasting ideal and mathematical number he says, "if all units are associable and without difference, we get mathematical number" (Metaphysics 1081a5-6). Then in the same passage thirteen

lines later he says, "mathematical number consists of undifferentiated units" (Metaphysics 1081a19). Thus, Plato indicates that these numbers are the referential objects of the state of dianoia (Republic 526a), and Aristotle unequivocally indicates that they are to be identified with mathematical number (i.e., mathematicals) in contrast to Ideal-Numbers. When these two points are combined with the abduction of mathematical objects in the Republic, we have what appears to be further support that Plato had a doctrine of mathematicals operative at this stage of his work, though he does not openly pronounce it in writing as such. But then does he ever do so in the dialogues? If Plato was to make such a dramatic change in later life, then why doesn't he indicate it in the last dialogues? There is no radical change. On the other hand, he does not give up the mathematicals later either. Thus the talk of the intermediate in the Philebus (15a-17a) and the role of the soul in the Laws (896e) does lend support to a doctrine of intermediate mathematicals. Nevertheless, there is no straightforward explicit statement regarding them in any of the dialogues.[20] On the other hand, Plato has not left the matter out of his dialogues entirely. Reference to them can be found in other middle dialogues besides the Republic. References occur in both the Phaedo and Timaeus. Of course, not all agree that the Timaeus is to be dated as a middle dialogue. However, irrespective of its dating, it does refer to the mathematicals. In my view, the more

critical later dialogues, including Theaetetus, Parmenides, and the Sophist, invoke a need for the intermediate mathematicals and the corresponding epistemological state to help solve some of the paradoxical problems posed therein. The crucial point is that the Republic inferences appear to provide very strong support for the hypothesis that Plato, at least as early as the Phaedo and Republic, had an underlying doctrine of intermediate mathematicals. Combining this evidence further with Plato's statements in the 7th Letter regarding disparagement of written doctrine, and the mathematical nature of his unwritten lecture, On the Good, it appears reasonable to abductively conjecture that this was in the main part of an unwritten, esoteric tradition within the Academy. Hence, Plato was following the procedure of silence of his earlier Pythagorean predecessors. But it is a doctrine not inconsistent with the subject-matter of the dialogues, Cherniss' arguments to the contrary, but rather a means of elucidating, clarifying, and possibly solving some of the perplexing problems contained therein.

Notes

[1]The Quadrivium was the four-fold Pythagorean division of mathematics into arithmetic, music, geometry, and spherics (or astronomy). Plato divided geometry into plane and solid, thereby making the Quadrivium into a Pentrivium.

[2]The exceptions are few. They include D'Arcy Wentworth Thompson and Gregory Des Jardins.

[3]Additionally, the Receptacle is identified as the Great and Small, Space, and the Bad where Good is realized.

[4]This will be seen to be central to the understanding of Plato's epistemology and ontology.

[5]This is still a problem in physics today, and is referred to as the "one body problem" and the "many body problem."

[6]Although it is clear that the members of the Academy were primarily mathematicans, with the possible exception of Aristotle who did not enjoy the the emphasis on mathematics.

[7]The great leaps of science are due to creative abduction initiated by the presence of anomalous phenomena or theoretical dilemmas.

[8]It is the combination of these two factors, analysis and the golden section, that is central to Plato's actual concerns.

[9]This shows how closely Aristotle agreed with Plato on this matter.

[10]It is probable that the reason Speusippus, rather than Aristotle, succeeded Plato, is because the former loved mathematics whereas the latter somewhat disparaged it.

[11]Mathematical numbers will generally be referred to as the intermediate "mathematicals" throughout the dissertation.

[12]Heath disputes this (1957).

[13]It is important to note that Proclus refers to analysis rather than division (dihairesis) as the finest method handed down. Some commentators, for example Gadamer (1980), place too much emphasis on division as being the essence of dialectic. To the contrary, it is an important tool to be employed in dialectic, but not a substitute for analysis. As Cherniss correctly notes, "[diahairesis] appears to be only an aid to reminiscence of the Ideas" (Cherniss, 1945, p. 55).

[14]In lectures I attended at the University of London in 1976.

[15]What follows sounds somewhat like an eastern doctrine of yoga. Of course, the Neoplatonic tradition maintains that Plato travelled widely, as Pythagoras had done before him.

[16]The Timaeus is generally considered late because it appears to make reference to problems already brought up in

the *Sophist*, a dialogue certainly later than the middle period group. My own view is that the *Timaeus* is late, and hence, shows that the mathematicals were maintained into Plato's later life.

[17]I take these dialogues to consist of problems presented to the reader that he might abduct the Platonic solutions.

[18]This is a very simple but powerful abduction.

[19]Any distinction is strictly the concoction of some commentators.

[20]These are merely allusions and inferences to them.

Birth of Plato, 429-427

<u>Charmides</u>
<u>Laches</u>
<u>Euthyphro</u>
<u>Hippias Major</u>
<u>Meno</u>

First visit to Sicily, 389-388
[Plato begins teaching at Academy, 388-387]

<u>?Cratylus</u>
<u>Symposium,</u> 385 or later

[Plato's initiation at Sais, Egypt at age of 47, 382-380]

<u>Phaedo</u>
<u>Republic</u>
<u>Phaedrus</u>
<u>Parmenides</u>
<u>Theaetetus,</u> 369 or later

[Aristotle enters Academy]
Second visit to Sicily, 367-366

<u>Sophistes</u>
<u>Politicus</u>

Third visit to Sicily, 361-360

<u>Timaeus</u>
<u>Critias</u>
<u>Philebus</u>
<u>Seventh Letter</u>
<u>Laws</u>
[<u>Epinomis</u>]

Death of Plato, 348-347

Figure # 1: Plato Chronology (after Ross, 1951, p. 10)

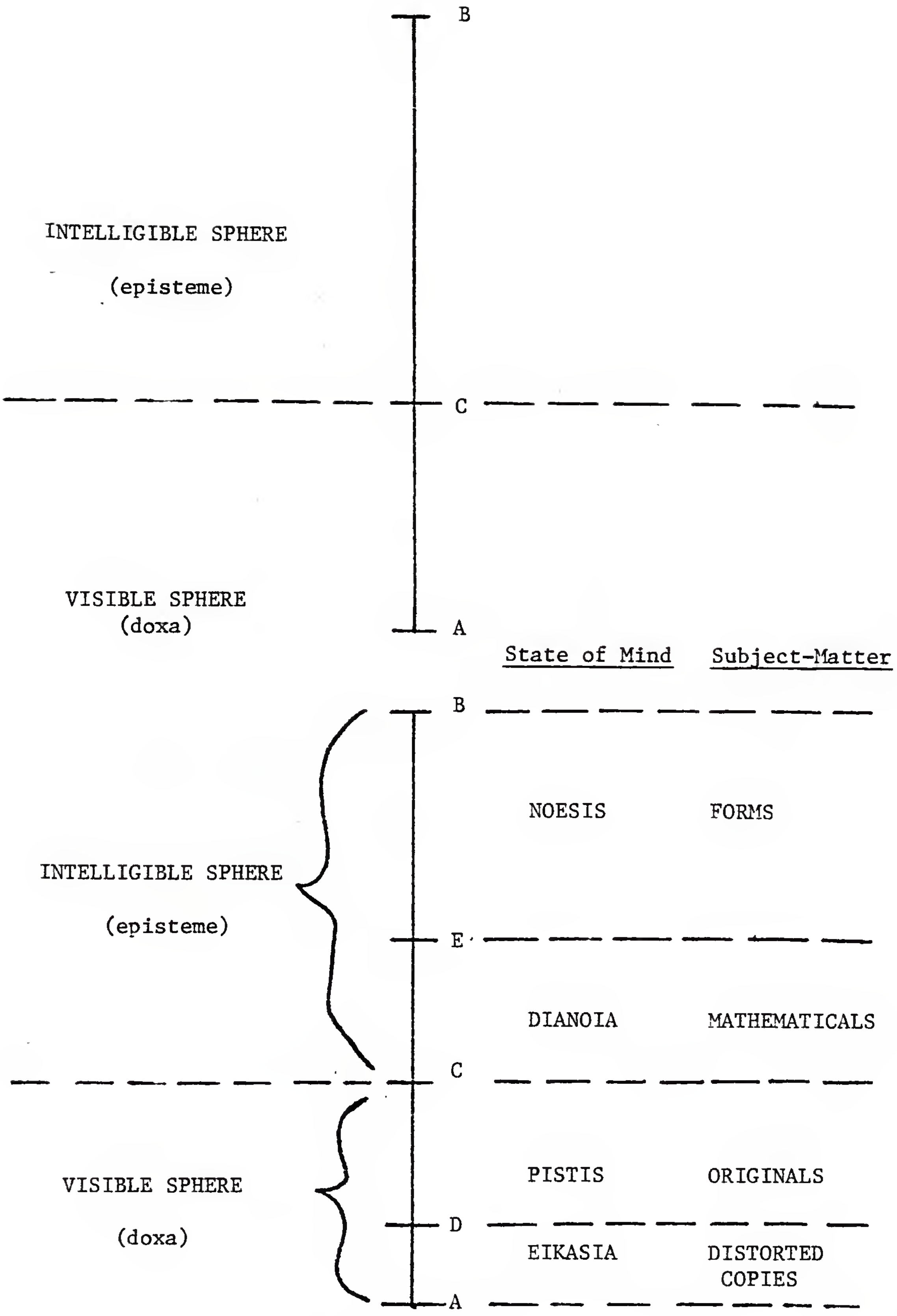


Figure # 2: Divided Line

CHAPTER IV

THE GOLDEN SECTION

Timaeus

Plato's doctrine of the intermediate nature of the soul is very closely related to his doctrine of the intermediate mathematicals with their corresponding state of dianoia. It is, in fact, the mathematicals embedded at this intermediate position in the soul that are infused into the bodily receptacle or space, providing it with number, ratio, and form. It is also the reason the philosopher can recollect, from within, the Forms. That is, the nature of the Idea-Numbers are embedded in the soul via the mathematicals. Hence, by turning to the mathematical disciplines, as shown in the *Republic*, the soul (or mind) is able to reawaken this knowledge, and ascends up the path of dianoia to noesis, where through a series of conversions it finally contemplates the Idea-Numbers, and ultimately the One Itself.

The dual functions of the mathematicals in the soul are then apparent. Ontologically, they solve the problem of how the sensible world participates in the intelligible world. The mathematicals in the world soul are infused into the sensible world. Each individual thing

participates in, and is assimilated toward, the Forms through this intermediate soul. Epistemologically, the mathematicals in the soul solve the problem of how we can come to know the Forms. It is through contemplation of the likenesses, i.e., the mathematicals within, that we are able to attain knowledge of the Idea-Numbers. This then is how the "worst difficulty" argument of the Parmenides is overcome.

In the Parmenides an argument is set forth which has the consequences of making it impossible for an individual to have knowledge of the Forms. The argument may be reconstructed as follows:

> I. Each Form has real Being just by itself.
> II. No such real Being exists in our world (of sensible things).
> III. These Forms have their Being in reference to one another, but not with reference to the sensible likenesses in our world.
> IV. Things in our world which bear the same names as the Forms, are related among themselves, but not to the Forms.
> V. Knowledge itself is knowledge of the real Being in Forms.
> VI. Knowledge in our world is knowledge of sensible things, and has no relation to Knowledge itself or the other Forms.
> Therefore, we cannot have knowledge of the Forms. (Parmenides 133a-134e)

The argument gains its strength on the view that sensible things and Forms are completely separate. But this only follows upon a superficial view of Plato's philosophy. It is dependent upon the completely bifurcated picture of the Platonic worlds of Being and Becoming. As Cornford points out, "it is this separation of the Forms

from their instances which threatens to isolate them in a world of their own, inaccesible to our knowledge" (Cornford, 1940, p. 95). But Plato hints that there may be something wrong with the premises of the argument.

> The worst difficulty will be this. . . . Suppose someone should say that the forms, if they are such as we are saying they must be, cannot even be known. One could not convince him that he was mistaken in that objection, unless he chanced to be a man of wide experience and natural ability, and were willing to follow one through a long and remote train of argument. Otherwise there would be no way of convincing a man who maintained that the forms were unknowable. (Parmenides 133b)

The essence of the solution revolves around the intermediate nature of the soul and mathematicals. However, the solution has already been suggested in the decidedly earlier middle dialogue, the Phaedo. There we find that man is "part body, part soul" (Phaedo 79b). The "soul is more like the invisible, and body more like the visible" (Phaedo 79b).

> The soul is most like that which is divine, immortal, intelligible, uniform, indissoluble, and ever self-consistent and invariable, whereas body is most like that which is human, mortal, multiform, unintelligible, dissoluble, and never self-consistent. (Phaedo 80b)

Thus, because man is part soul and part body, it follows that man has access to the Forms via the soul. Thus, the premises of complete separation of the intelligible and sensible worlds are suspect.

In the Timaeus the doctrine of the intermediate soul is established more fully. "And in the center [of the

Cosmos] he put the soul, which he diffused throughout the body" (Timaeus 34b). Thus, the entire body of the Cosmos participates in the Forms through the world soul which is "diffused throughout" it.

> [Deity] made the soul in origin and excellence prior to and older than the body, to be the ruler and mistress, of whom the body was to be the subject. . . . From the being which is indivisible and unchangeable [i.e., ousia], and from that kind of being which is distributed among bodies [i.e., genesis] he compounded a third and intermediate kind of being. He did likewise with the same and the different, blending together the indivisible kind of each with that which is portioned out in bodies. Then taking the three new elements, he mingled them all into one form, compressing by force the reluctant and unsociable nature of the different into the same. When he had mingled them with the intermediate kind of being and out of three made one, he again divided this whole into as many potions as was fitting, each portion being a compound of the same, the different, and being. (Timaeus 34c-35b)

Thus, the soul in its intermediate position is compounded out of the material of both the intelligible and sensible worlds. As such it touchs, so to speak, both of the worlds. Because of this the individual has access to knowledge of the Forms. This process of assimilation to the higher knowledge begins to occur at the intermediate state of awareness of dianoia, and the contemplation of the mathematicals. "A silent inner conversation of the the soul with itself, has been given the special name of dianoia" (Sophist 263d). Dialectic then takes one through noesis to the direct contemplation of the Forms.

However, besides the epistemological role of dianoia in relation to the intermediate soul and the mathematicals, the soul and mathematicals also play a very important ontological role. The soul is responsible for the infusion of number, ratio, and proportion into the bodily receptacle of space. My argument is that the soul infuses the mathematicals as copies of the Forms, into the bodily receptacle of the Cosmos, producing ordered extension, ordered movement, and the various qualities related to number. This is how the sensible particulars participate in the Forms. It is through the mathematicals infused by the soul into the visible world to give it form.

Ross, at one point in his work, recognized this intermediate role of the soul and its mathematicals.

> It would be a mistake to describe Plato as having . . . at any stage of his development, made a complete bifurcation of the Universe into Ideas and sensible things. For one thing we have the casual reference to "equals themselves" [Phaedo 74c]--an allusion to mathematical entities which are neither ideas nor sensible things, an allusion which paves the way for the doctrine of the Intermediates. (Ross, 1951, pp. 25-26)

There is an additional suggestion of mathematicals in the Timaeus. There referring to the receptacle or space, Plato says:

> She is the natural recipient of all impressions, and is stirred and informed by them, and appears different from time to time by reason of them. But the forms which enter into and go out of her are the likeness of eternal realities modeled after their

> patterns in a wonderful and mysterious manner, which we will hereafter investigate. (Timaeus 50c)

Ross also sees this passage as a reference to mathematicals. He explains that for Plato sensible things were moulded in space, "by the entrance into it of shapes which are likensses of the eternal existents, the Forms" (Ross, Aristotle's Metaphysics, vol. 1, pp. 167-168). Cherniss, of course, takes issue with this interpretation. He claims that Ross, "mistakenly takes ta eisionta kai exionta of Timaeus 50c to be a class distinct from sensibles as well as Ideas" (Cherniss, 1945, p. 76, fn.79). But this must be argued by Cherniss because he has rejected Aristotle's remarks about Plato's intermediate mathematicals doctrine as mere misinterpretation and misunderstanding on Aristotle's part. To the contrary, on my view, this passage is very suggestive of Plato's underlying mathematical doctrine.

Proportion

What then were these mathematicals infused via the world soul into the spatial receptacle? The answer is ratios (logos) and proportions (analogia). This was the Pythagorean conception of Plato, that the order and harmony of number was infused into the sensible world.

> [Deity] made the body of the universe to consist of fire and earth. But two things cannot be rightly put together without a third; there must be some bond of union between them. And the fairest bond is that

> which makes the most complete fusion of itself and the things which it combines, and [geometrical] proportion is best adapted to effect such a union. For whenever in any three numbers, whether cube or square, there is a mean, which is to the last term what the first term is to it, and again, when the mean is to the first term as the last term is to the mean--then the mean becoming first and last, and the first and last both becoming means, they will all of them of necessity come to be the same, and having become the same with one another will be all one. (_Timaeus_ 31b-32a)

What Plato is referring to is none other than a geometrical proportion. He is depicting the relation A:B::B:C. As an example we might have, 1:2::2:4, or alternatively, 1:3::3:9. This is the kind of proportion he feels is best adapted to be the bond of union making the Cosmos one. But this single mean is sufficient only for plane figures or square numbers. If one proceeds to solid figures and cube numbers, then two means are required.

> If the universal frame had been created a surface only and having no depth, a single mean would have sufficed to bind together itself and the other terms, but now, as the world must be solid, and solid bodies are always compacted not by one mean but by two, God placed water and air in the mean between fire and earth, and made them to have the same proportion so far as was possible--as fire is to air so is air to water, and as air is to water so is water to earth--and thus he bound and put together a visible and tangible heaven. And for these reasons, and out of such elements which are in number four, the body of the world was created, and it was harmonized by proportion. (_Timaeus_ 32a-c)

Thus, the resulting continued geometric proportion has the following relation: fire : air :: air : water :: water : earth. Expressing this numerically we have:

1:2::2:4::4:8. Eight is of course a cube number, being the cube of 2. Alternatively, we could have: 1:3::3:9::9:27. Again we end with a solid as 27 is the cube of 3. Geometric proportion was considerd to be the primary proportion because the other proportions (i.e., harmonic and arithmetic) require it, but it does not require them. Thus Corford writes,

> [Adrastus] says that geometrical proportion is the only proportion in the full and proper sense and the primary one, because all the others require it, but it does not require them. The first ratio is equality (1/1), the element of all other ratios and the proportions they yield. He then derives a whole series of geometrical proportions from "the proportion with equal terms" (1,1,1) according to the following law: "given three terms in continued proportion, if you take three other terms formed of these, one equal to the first, another composed of the first and the second, and another composed of the first and twice the second and the third, these new terms will be in continued proportion." (Cornford, 1956, pp. 47-48)

The double and triple proportions that Plato establishes in the world soul can be seen immediately to follow from this formula. Thus, we get 1 for the first term, 1+1=2 for the second term, and 1+2+1=4 for the third term. This gives the proportion of 1:2:4, or the double proportion through the second power or square. To extend the double proportion further into the third power or cube number, we simply multiply the resulting second and third terms, 2x4=8.

In a similar manner Adrastus' formula can be employed to arrive at Plato's triple proportion, by beginning with

the first three units of the double proportion (i.e., 1,2,4). Thus, 1 is the first term, 1+2=3 is the second term, and 1+(2x2)+4=9 is the third term. This gives the triple proportion through the second power or square number, 1:3:9. By multiplying the resulting second and third terms (3x9), we arrive at the third power or cube number, 27. Hence, we arrive at Plato's triple proportion of 1:3:9:27.

The doctrines of the Timaeus are clearly a perpetuation of Pythagorean doctrine.

> It is well known that the mathematics of Plato's Timaeus is essentially Pythagorean. . . . [in passage 32a-b] by planes and solids Plato certainly meant square and solid numbers respectively, so that the allusion must be to the theorems established in Eucl. VIII, 11, 12, that between two square numbers there is one mean proportional number and between two cube numbers there are two mean proportionals. (Heath, 1956, vol. 2, p. 294)

Then the world soul which has been compounded out of being and becoming is divided into a harmonious relation very much along the same lines as Plato mentioned earlier regarding the proportions of the 4 elements (Timaeus 31b-32c). However, it is left to the reader to make the connection (abductive leap) between the geometric proportion of the four elements at Timaeus 32a-c, on the one hand, and the proportionate division of the world soul at Timaeus 35b-36b, on the other.

> And he [Deity] proceeded to divide after this manner. First of all, he took away one part of the whole [1], and then he separated a second part which was double the first [2], and then he took away a third part which was

> half as much again as the second and three times as much as the first [3], and then he took a fourth part which was twice as much as the second [4], and a fifth part which was three times the third [9], and a sixth part which was eight times the first [8], and a seventh part which was twenty-seven times the first [27]. After this he filled up the double intervals [that is, between 1,2,4,8] and the triple [that is, between 1,3,9,27], cutting off yet other portions from the mixture and placing them in the intervals, so that in each interval there were two kinds of means, the one exceeding and exceeded by equal parts of its extremes [as for example, 1, 4/3, 2, in which the mean 4/3 is one third of 1 more than 1, and one third of 2 less than 2], the other being that kind of mean which exceeds and is exceeded by an equal number. Where there were intervals of 3/2 [a fifth] and of 4/3 [fourth] and of 9/8 [tone], made by the connecting terms in the former intervals, he filled up all the intervals of 4/3 with the interval of 9/8, leaving a fraction over, and the interval which this fraction expressed was in the ratio of 256 to 243. And thus the whole mixture out of which he cut these portions was all exhausted by him. (Timaeus 35b-36b)

In this manner Plato has placed in the soul the odd and even numbers, and their perfect relationships in terms of ratio and proportion. In this way he has included the harmonic and arithmetic means within the primary geometric proportion. Thus, in terms of the even numbers we have the following geometric proportion, embodying two geometric means and ending in a solid number: 1, 4/3(harmonic), 3/2(arithmetic), 2, 8/3, 3, 4, 16/3, 6, 8. In this series, 1,2,4, and 8, are in continued geometric proportion. Likewise, the odd numbers find their expression as follows: 1, 3/2, 2, 3, 9/2, 6, 9, 27/2, 18, 27. Here the numbers, 1,3,9, and 27, are in continued geometric proportion. In

this way the soul comprehends within itself the fundamental proportions of the Cosmos. Furthermore, the soul, by being infused into the bodily receptacle, provides sensible things with number, ratio, and proportion. These numbers cover a range of 4 octaves and a major 6th. It is of note that the major 6th will be seen to involve a Fibonacci approximation to the golden section. As Adrastus puts it,

> Plato is looking to the nature of things. The soul must be composed according to a harmonia and advance as far as solid numbers and be harmonised by two means, in order that extending throughout the whole solid body of the world, it may grasp all the things that exist. (Cornford, 1956, p. 68)

It is in this way that the body of the world was "harmonized by proportion" (*Timaeus* 32c). But these numbers in the soul are linked to the proportional relationship of the four elements, which he has alluded to earlier (*Timaeus* 31b-32c). To this point we have only considered the proportions relating to commensurable numbers. Does Plato have a similar embodiment of the golden section in the *Timaeus* as I have claimed he does in the Divided Line of the *Republic*? Let us turn first for a moment to a consideration of the *Epinomis*.

Taylor & Thompson on the *Epinomis*

This argument will require a somewhat long and circuitous route. However, it will be valuable in aiding the strength of my position concerning the relevance of the golden section to Plato's philosophy of number and

proportion. I will be considering, in particular, the arguments of A.E. Taylor and D.W. Thompson. In the Epinomis Plato suggests that "the supreme difficulty is to know how we are to become good men [people]" (Epinomis 979c). Furthermore, it is the good man who is happy. To be good, a soul must be wise (Epinomis 979c). Thus Plato asks, "What are the studies which will lead a mortal man to wisdom" (Epinomis 973b)? The answer is "the knowledge of number" (Epinomis 976e). He then argues that if one were utterly unacquainted with number, it would be impossible to give a rational account of things. But if no rational account is possible, then it is impossible to be wise. But if not wise, then it is impossible to become perfectly good. And if one is not perfectly good, then he is not happy (Epinomis 977c-d).

> Thus there is every necessity for number as a foundation . . . all is utterly evacuated, if the art of number is destroyed. (Epinomis 977d-e).

This passage from the Epinomis appears to contain echoes of the Pythagorean Philolaus.

> And all things that can be known contain number; without this nothing could be thought or known. (Philolaus in Kirk and Raven, 1975, p. 310)

It is therefore evident to the very last dialogue that Plato is essentially a Pythagorean.

It should be noted here that I take the Epinomis to be one of Plato's works. Even it it were true that it was edited or written by Plato's pupil Philip, it nevertheless

is Platonic doctrine. Taylor brings this out when he says:

> . . . my own conviction is that it is genuine and is an integral part of the Laws. Those who have adopted, on the slenderest of grounds, the ascription to Philippus of Opus [or of Medma] at least recognise that the author is an immediate scholar of Plato, specially competent in mathematical matters, and that the work was issued from the first along with the Laws. . . . We may accept the matter of a mathematical passage from the dialogue as genuinely Platonic with reasonable confidence. (Taylor, 1926, p. 422)

Now the "art of number" referred to at Epinomis 977e is not limited to knowledge of the commensurable integers. The philosopher must also attain knowledge of the incommensurables. Referring now to the Thomas translation of the Epinomis:

> There will therefore be need of studies [mathamata]: the first and most important is of numbers in themselves, not of corporeal numbers, but of the whole genesis of the odd and even, and the greatness of their influence on the nature of things. When the student has learnt these matters there comes next in order after them what they call by the very ridiculous name of geometry, though it proves to be an evident likening, with reference to planes, of numbers not like one another by nature; and that this is a marvel not of human but of divine origin will be clear to him who is able to understand. (Epinomis 990c-991b, Thomas, 1957, vol. 1, p. 401)

Again there is the cryptic suggestion that he who has assimilated his mind near to deity, will be able to apprehend the deeper meaning involved. The reference to "numbers not like one another by nature" is to incommensurables. As Thomas has pointed out:

> The most likely explanation of "numbers not like one another by nature" is "numbers

> incommensurable with each other"; drawn as two lines in a plane, e.g. as the side and diagonal of a square, they are made like to one another by the geometer's art, in that there is no outward difference between them as there is between an integer and an irrational number. (Thomas, 1957, vol. 1, p. 401, fn.b)

Taylor places great emphasis on this passage from the Epinomis. He feels that it supplies a clue as to why Plato made the apeiron (the infinite or space) of the Pythagoreans into an indefinite dyad (or greater and smaller). In the Philebus (24a ff.), Plato has Socrates use the old Pythagorean antithesis of apeiron and peras (limit or bound). However, Aristotle indicates that the apeiron came to be the greater and smaller for Plato. As Taylor says,

> Milhaud, Burnet and Stenzel are all, rightly . . . agreed, that the Platonic formula is somehow connected with the doctrine of "irrational" numbers. . . . Again, when Milhaud, Burnet, Stenzel, all look for the explanation of the formula in the conception of the value of an "irrational," they are plainly on the right track, as the passage of the Epinomis [990c-991b] . . . demonstrates. (Taylor, 1926, p. 421)

They were on the right track but they did not go deep enough in their interpretations. Taylor does go deeper:

> When Plato replaced the Pythagorean apeiron by the "duality" of the "great and small," he was thinking of a specific way of constructing infinite convergent series which his interpreters seem not to have identified. (Taylor, 1926, p. 422)

The fifth century B.C. philosophers had been concerned with the value of the square root of 2, whereas the fourth century B.C. was concerned with the Delian problem and the

cube root of 2. The method which was employed, and alluded to in the Epinomis, was the technique of making increasingly more accurate rational approximations to the number's irrational value. As Taylor points out,

> . . . there is a general rule given by Theon of Smyrna (Hiller, p. 43f.) for finding all the integral solutions to the equation, or, as the Greek expression was, for finding an unending succession of "rational diameters," that is, of increasingly accurate rational approximations to the square root of 2, the "ratio of the diagonal to the side." The rule as given by Theon is this. We form two columns of integers called respectively "sides" and "diagonals." In either column we start with 1 as the first term; to get the rest of the "sides," we add together the nth "side" and the nth "diagonal" to form the (n+1)th "side";in the column of "diagonals," the (n+1)th "diagonal" is made by adding the nth "diagonal" to twice the nth side. Fortunately also Proclus . . . has preserved the recognised demonstration of this rule; it is a simple piece of geometry depending only on the identity $(a+b)^2 + b^2 = 2(a/2)^2 + 2(a/2+b)$ which forms Euclid's proposition II, 10. (Taylor, 1926, pp. 428-429)

In effect by dividing the diagonal number by the side number, one obtains successively closer approximations to, or a "rational value" for, the square root of 2. Some interesting features emerge when one considers the side and diagonal method of approximation. First is the fact that at each step of convergence, the new convergent term is nearer to the desired limit value. Second, the convergents alternate in being "rather less and rather greater than the value to which they are approximations" (Taylor, 1926, p. 430). To illustrate, 7/5 is less than the square root of 2, since $7^2 = 2x5^2 - 1$; 17/12 is greater than the square

root of 2, $17^2 = 2 \times 12^2 + 1$ (Taylor, 1926, p. 430). Third,

> . . . the interval, or absolute distance, between two successive "convergents" steadily decreases, and by taking n sufficiently large, we can make the interval between the nth and (n+1)th convergent less than any assigned rational fraction s, however small, and can therefore make the interval between the nth convergent and the required "irrational" smaller still than s. (Taylor, 1926, p. 431)

Fourth, ". . . the method is manifestly applicable to any 'quadratic' surd, since it rests on the general formula $(\sqrt{a}-b) = (a-b^2)/(\sqrt{a}+b)$" (Taylor, 1926, p. 431). Fifth,

> . . . we are not merely approximating to a "limit," we are approximating to it from both sides at once; $\sqrt{2}$ is at once the upper limit to which the series of values which are too small, 1, 7/5, . . . are tending, and the lower limit to which the values which are too large, 3/2, 17/12, . . . are tending. This, as it seems to me, is manifestly the original reason why Plato requires us to substitute for the apeiron as one thing, a "duality" of the great and small. $\sqrt{2}$ is an apeiron, because you may go on endlessly making closer and closer approximations to it without ever reaching it; it never quite turns into a rational number, though it seems to be on the way to do so. But also it is a "great and small" because it is the limit to which one series of values, all too large, tends to decrease, and also the limit to which another series, all too small, tends to increase. (Taylor, 1926, pp. 431-432)

On this view Taylor concludes that the meaning of the Epinomis passage (referred to) is to evaluate all quadratic surds, and in so doing discover their appropriate side and diagonal converging series. D.W. Thompson accepts Taylor's view to this point and then elaborates upon it. Thompson argues that "ho arithmos" is to be taken in its technical sense as surd.

> . . . ho arithmos may be used here in its technical sense, meaning a surd or "irrational number," especially $\sqrt{2}$; and the general problem of Number may never have been in question at all. It was the irrational number, the numerical ratio (if any) between two incommensurable segments, which was a constant object of search, whose nature as a number was continually in question, and whose genesis as a number cried aloud for explanation or justification. (Thompson, 1929, pp. 43-44)

Thompson's contribution is to go beyond Taylor's insight into the nature of the great and small. Thompson shows how the One acted as an equalizer relative to the greater and smaller.

> . . . that the side and diagonal numbers show us what Plato means by the Great-and-Small, or Aristotle by his Excess-and-Defect, is certain; Prof. Taylor [Taylor 1926 & 1927] has made it seem clear and obvious. But Prof. Taylor has not by any means made it clear what Plato meant by τὸ ἕν . . . . it is another name for that Unit or "Monad" which we continually subtract from the "Great" or add to the "Small," and which so constructs for us the real number. (Thompson, 1929, p. 47)

This will become more evident by examining the side and diagonal numbers for the approximation to $\sqrt{2}$.

Sides	Diagonals
1	1
2	3
5	7
12	17

etc.

What Thompson noticed was that

> the striking and beautiful fact appears that this "excess or defect" is always capable of being expressed by a difference of 1. The square of the diagonal number (i.e., of what

> Socrates calls the "rational diagonal" [rational diameter, Republic 546c] is alternately less or more by one than the sum of the square of the sides." (Thompson, 1929, p. 46)

This may be depicted in the following table

$$2 \times 1^2 = 1^2 + 1$$

$$2 \times 2^2 = 3^2 - 1$$

$$2 \times 5^2 = 7^2 + 1$$

$$2 \times 12^2 = 17^2 - 1$$

etc.

Thus, the alternating excess and defect is in each case measured by one unit. Taylor continues:

> For Unity then comes into the case in a twofold capacity. It is the beginning archē [ἀρχή] of the whole series. Then again as the series proceeds, the "One" has to be imported into each succeeding Dyad, where it defines (ὁρίζει) the amount of excess or defect, and equates or equalises (ἰσάζει) the two incompatible quantities. (Thompson, 1929, pp. 46-47)

Thus, Thompson sees "the One as the continual 'equaliser' of the never-ending Dyad" (Thompson, 1929, p. 50). The use of side and diagonal numbers were therefore employed by the Greeks to gain rational approximations to the value of surds. It is true that the successive convergents never end. However, the difference between successive convergents can be made to be less than any predetermined value. Taylor writes:

> They never actually meet, since none of the "convergents" is ever the same as its successor, but, by proceeding far enough with the series we can make the interval between two successive "convergents" less than any assigned difference, however small. (Taylor quoted in Thompson, 1929, p. 47)

The use of side and diagonal numbers appears to have been central to the Greek quest for a rational account of irrational numbers. The relation of the One and Indefinite Dyad on the hybrid Taylor-Thompson account does much to clarify the matter. However, Thompson proceeds further. He argues that there is another table "which may be just as easily, or indeed still more easily derived from the first, and which is of very great importance" (Thompson, 1929, p. 50). But there is no mention of it in the history of Greek mathematics. My view is that the reason there is no mention of it is because it lay at the center of the early unwritten Pythagorean tradition, at a time when silence was taken very seriously.

> We remember that, to form our table of side and diagonal numbers, we added each side-number to its own predecessor, that is to say, to the number standing immediately over it in the table, and so we obtained the next diagonal; thus we add 5 to 2 to get 7. (Thompson, 1929, pp. 50-51)

Hence we would get the following $\sqrt{2}$ table:

side	diagonal
1	1
2	3
5	7
12	17

etc.

But now instead of adding the two side numbers 5 and 2 to get the diagonal 7, we add each side number to the previous diagonal to get the new diagonal. Thus, for example, we add 5 to 3 to get 8. The new side numbers are arrived at in the same manner as the previous method.

However, we now come up with a very interesting table.

1	1
2	3
5	8
13	21
34	55
89	144

etc.

This is none other than the Fibonacci series. This is

> the famous series . . . supposed to have been "discovered" or first recorded by Leonardo of Pisa, nicknamed the Son of the Buffalo, or "Fi Bonacci." This series has more points of interest than we can even touch upon. It is the simplest of all additive series, for each number is merely the sum of its two predecessors. It has no longer to do with sides and diagonals, and indeed we need no longer write it in columns, but a single series, 1,1,2,3,5,8,13,21,etc. (Thompson, 1929, p. 51)

But Thompson the morphologist saw even more deeply into the significance of this series. Thus he said:

> It is identical with the simplest of all continued fractions. . . . Its successive pairs of numbers, or fractions, as 5/3, 8/5, etc., are the number of spirals which may be counted to right and to left, on a fir-cone or any other complicated inflorescence. . . . But the main property, the essential characteristic, of these pairs of numbers, or fractions, is that they approximate rapidly, and by alternate excess and defect, to the value of the Golden Mean. (Thompson, 1929, pp. 51-52)

Thus far Thompson has penetrated into the Platonic arcanum. The positive value of the golden section is $(\sqrt{5}+1)/2$. This incommensurable, placed in abbreviated decimal notation is 1.6180339.... If we take the Fibonacci series, 1,1,2,3,5,8,13,21,34,55,89,144,233,etc. and form a

fraction out of each set of two succeeding terms, placing the first term in the denominator, and the second term in the numerator, we get the following series of fractions: 1/1,2/1,3/2,5/3,8/5,13/8,21/13, etc. When looking at the corresponding decimal values, note that the first and succeeding alternate terms are deficient, the second and succeeding even terms are in excess. The terms of the series asymptotically continue to get closer to the limit value of the golden section. The values are 1, 2, 1.5, 1.666..., 1.6, 1.625, 1.6153846..., 1.6190476..., 1.617647..., 1.6181818..., 1.6179775..., 1.6180555.... Thus, by the time we arrive at the value of 233/144, 1.6180555..., we are very near to the value of the golden section. In other words, these fractions derived from this simplest of additive series, the Fibonacci series, asymptotically converge upon the golden cut in the limit.

I will again quote Thompson extensively because I believe that he has penetrated deeply into the mystery.

> The Golden Mean itself is, of course, only the numerical equivalent, the "arithmetisation," of Euclid [Elements] II.11; where we are shown how to divide a line in "extreme and mean ratio," as a preliminary to the construction of a regular pentagon: that again being the half-way house to the final triumph, perhaps the ultimate aim, of Euclidean or Pythagorean geometry, the construction of the regular dodecahedron, Plato's symbol of the Cosmos itself. Euclid himself is giving us a sort of algebraic geometry, or rather perhaps a geometrical algebra; and the [Fibonacci] series we are now speaking of "arithmeticises" that geometry and that algebra. It is surely much more than coincidence that this [Fibonacci] series is

> closely related to Euclid II.11, and the other [$\sqrt{2}$ series] (as Theon expressed it), to the immediately preceding proposition [Euclid II.10]. (Thompson, 1929, p. 52)

It is my contention that it certainly is not a coincidence for Euclid II.10 and 11 to be grouped together, and further for each of these propositions to correspond respectively to the $\sqrt{2}$ and Fibonacci series. This is the Pythagoreanism of Plato extended through the writings of one of his Academic descendents, Euclid. And it is not insignificant that the 11th proposition of Book II of Euclid is preparatory to the two dimensional construction of the pentagon, and ultimately to the three dimensional construction of the embodiment of the binding proportion of the Cosmos itself, the dodecahedron.

> Since the students of the history of Greek geometry seem agreed that the contents of Euclid II are all early Pythagorean, there is no reason why the rule given by Theon [side and diagonals for $\sqrt{2}$] should not have been familiar not only to Plato, but to Socrates and his friends in the fifth century. The probability is that they were acquainted with it, and thus knew how to form an endless series of increasingly close approximations to one "irrational," $\sqrt{2}$. (Taylor, 1926, p. 429)

But by extension of this argument regarding the early Pythagorean contents of Book II of Euclid's Elements, the case of II.11, is covered as well. It is also early Pythagorean in origin. As such it was part of the esoteric, unwritten tradition. The so-called discovery by Leonardo of Pisa regarding the Fibonacci series is at best a rediscovery. And most probably he is given credit

because he is the first to expose openly the esoteric tradition by putting it in writing.

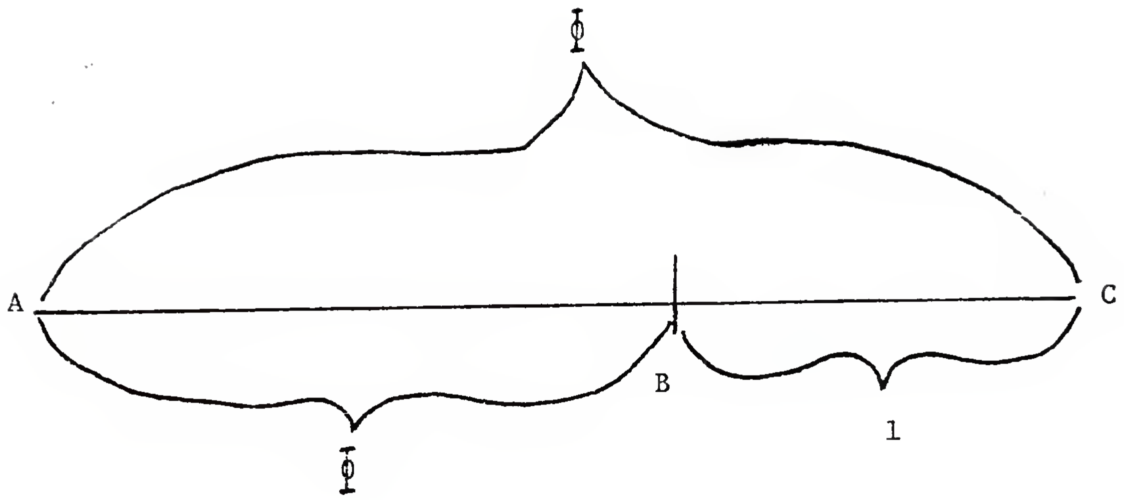
In Euclid II.11, we have a line AC divided in extreme and mean ratio at B. See Figure # 3, p. 147. We thus have the three magnitudes AC, AB, and BC. The relations are such that $(AB)^2 = AC \times BC$. From the Fibonacci series we can replace the geometrical magnitudes with any three consecutive numbers from the series. As Thompson points out:

> . . . the square of the intermediate number . . . [is] equivalent--approximately equivalent--to the product of the other two. Observe that, precisely as in the former case [i.e., side and diagonal numbers for 2], the approximation gets closer and closer; there is alternate excess and defect; and (above all) the "One" is needed in every case, to equate the terms, or remedy the defective approximation. (Thompson, 1929, p. 52)

Thus the "One" plays the role of modifying the excess and deficiency of the successive squares. This will become more clear through the following illustration. Where AC=21, AB=13, and BC=8, we have the approximate proportion of 21:13::13:8. But 21/13 = 1.6153846..., and 13/8 = 1.625. Hence the relation is not exact. How then does the "one" act as an equalizer in the view of Thompson? Taking the formula $AB^2 = AC \times BC$, we get the following:

$$13^2 = (21 \times 8) + 1$$
$$169 = 168 + 1$$

Again we can substitute any three consecutive numbers of the Fibonacci series. In each case we will discover that the "One" acts as an equalizer. But again it does so



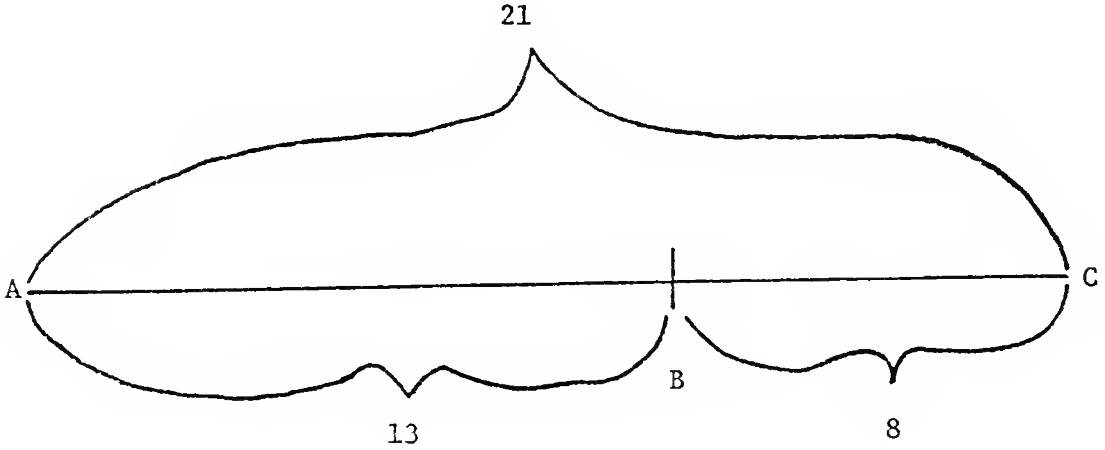


Figure # 3: Golden Cut & Fibonacci Approximation

through the equalizing alternately of a unit of excess and a unit of deficiency.

$$3^2 = (5x2)-1$$
$$5^2 = (8x3)+1$$
$$8^2 = (13x5)-1$$
$$13^2 = (21x8)+1$$

etc.

It is in this manner that we discover more meaning in the latent suggestions in the *Epinomis* of the method of the geometer's art in discovering the nature of surds or incommensurables. There are echoes of it in the *Meno* and the Theaetetus. And it is not a coincidence that these Pythagorean doctrines were later published in sequence (i.e., *Elements,* Book II.10 & 11) by the Platonic disciple, Euclid.

> It is inconceivable that the Greeks should have been unacquainted with so simple, so interesting and so important a series [Fibonacci series]; so closely connected with, so similar in its properties to, that table of side and diagonal numbers which they knew familiarly. Between them they "arithmeticise" what is admittedly the greatest theorem, and what is probably the most important construction, in all Greek geometry. Both of them hark back to themes which were the chief topics of discussion among Pythagorean mathematicians from the days of the Master himself; and both alike are based on the arithmetic of fractions, with which the early Egyptian mathematicians, and the Greek also, were especially familiar. Depend upon it, the series which has its limit in the Golden Mean was just as familiar to them as that other series whose limit is $\sqrt{2}$. (Thompson, 1929, p. 53)

This provides further evidence that Plato and the other Pythagoreans of the time were familiar with the golden section. But we must inquire more fully to see just what significance the golden section may have had for Plato's philosophy.

ϕ and the Fibonacci Series

Why then might Plato feel that the golden section was so important? We shall look at some of the properties of the golden mean for the answer.

> The "Golden Section," $(1+\sqrt{5})/2=1.6180339...$, positive root of the equation $x^2=x+1$, has a certain number of geometrical properties which make it the most remarkable algebraical number, in the same way as π and e are the most remarkable transcendendent numbers. (Ghyka, 1946, p. 2)

Here are some of the golden section's properties.

$\phi = (\sqrt{5}+1)/2 = 1.6180339875....$

1.618... is a very accurate approximation to ϕ.

$1/0 = (\sqrt{5}-1)/2 = 0.618... = \phi-1$

$\phi^2 = (\sqrt{5}+3)/2 = 2.618... = \phi+1$

$\phi^3 = \phi^2+\phi$

$\phi = \phi + \phi$

The geometrical series known as the "golden series," in which the golden section ratio is the fundamental module, $1, \phi, \phi^2, \phi^3, \phi^4, ... \phi^n$, possesses the remarkable property of being at the same time both additive and geometrical. It is an additive series in that each term is the sum of the two preceding terms, for example $\phi^3 = \phi^2 + \phi$.

The Fibonacci series

> tends asymptotically towards the ϕ progression with which it identifies itself very quickly; and it has also the remarkable property of producing "gnomonic growth" (in which the growing surface or volume remains homothetic, similar to itself) by a simple process of accretion of discrete elements, of integer multiples of the unit of accretion, hence the capital role in botany of the Fibonacci series. For example, the fractionary series 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55, 55/89, 89/144,... appears continually in phyllotaxis (the section of botany dealing with the distribution of branches, leaves, seeds), specially in the arrangements of seeds. A classical example is shown in the two series of intersecting curves appearing in a ripe sunflower (the ratios 13/21,21/34,34/55, or 89/144 appear here, the latter for the best variety). The ratios 5/8,8/13, appear in the seed-cones of fir-trees, the ratio 21/34 in normal daisies. If we consider the disposition of leaves round the stems of plants, we will find that the characteristic angles or divergencies are generally found in the series 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34, 21/55,.... (Ghyka, 1946, pp. 13-14)

As seen earlier, the golden section ratio existing between the two parts of a whole, such as the two segments of a line, determines between the whole and its two parts a continuous geometrical proportion. The ratio between the whole to the greater part is equal to the ratio between the greater to the smaller part. We saw this earlier in the Divided Line. Thus, the simplest asymmetrical section and the corresponding continuous proportion is that of the golden section. Of course the successive terms of the Fibonacci series can be employed as an approximation to the golden cut.

To illustrate and review, the typical continuous geometrical proportion is a:b::b:c, in which b is the proportional (geometrical mean) between a and c. In the geometrical proportion b becomes the "analogical invariant which besides the measurement brings an ordering principle" (Ghyka, 1946, p. 2). This analogical invariant transmits itself through the entire progression as a characteristic ratio, acting as a module.

But one can obtain a greater simplification by reducing to two the number of terms, and making c=a+b. Thus, taking the Divided Line, where the two segments are designated a and b, and the whole line is of length c, if they are fit into the formula, then the continuous proportion becomes: a:b::b:(a+b). This may be translated into $(b/a)^2=(b/a)+1$. If one makes b/a=x, it is quickly seen that x, positive root of the equation $x^2=x+1$, is equal to $(\sqrt{5}+1)/2$, the golden section. Thus, the golden section is the most logical and therefore the simplest asymmetrical division of a line into two segments. This result may also be found by employing "Ockham's Razor," for which the reader may consult Ghyka's work (Ghyka, 1946, p. 3).

It is interesting to note that the golden section (and series), and its corresponding approximation, the Fibonacci series, are found throughout nature.

> The reason for the appearance in botany of the golden section and the related Fibonacci series is to be found not only in the fact that the ϕ series and the Fibonacci series

> are the only ones which by simple accretion, by additive steps, can produce a "gnomonic," homothetic, growth (these growths, where the shapes have to remain similar, have always a logarithmic spiral as directing curve), but also in the fact that the "ideal angle" (constant angle between leaves or branches on a stem producing the maximum exposition to vertical light) is given by [the formula] $(a/b)=b/(a+b)$, $(a+b)=360°$; one sees that b divides the angular circumference (360°) according to the golden section. $b=360°/\phi = 220° 29' 32"$; $a = b/\phi = 137° 30' 27" 95$. The name of "ideal angle" was given to "a" by Church, who first discovered that it corresponds to the best distribution of the leaves; the mathematical confirmation was given by Wiesner in 1875. (Ghyka, 1946, pp. 14-16)

The dynamic growth in nature based on the golden section and Fibonacci series is evident in the logarithmic spirals (see Figure # 4, p. 153), so preponderant in plants, insects, and animals. As Thompson has pointed out, it is sometimes difficult to pick out these logarithmic spirals, or what he terms genetic spirals. However, like the clues scattered throughout Plato's works, they are there for those with eyes to see.

> It is seldom that this primary, genetic spiral catches the eye, for the leaves which immediately succeed one another in this genetic order are usually far apart on the circumference of the stem, and it is only in close-packed arrangements that the eye readily apprehends the continuous series. Accordingly in such a case as a fir-cone, for instance, it is certain of the secondary spirals or "parastichies" which catch the eye; and among fir-cones, we can easily count these, and we find them to be on the whole very constant in number. . . . Thus, in many cones, such as those of the Norway spruce, we can trace five rows of scales winding steeply up the cone in one direction, and the three rows winding less steeply the other way; in certain other species, such as the common

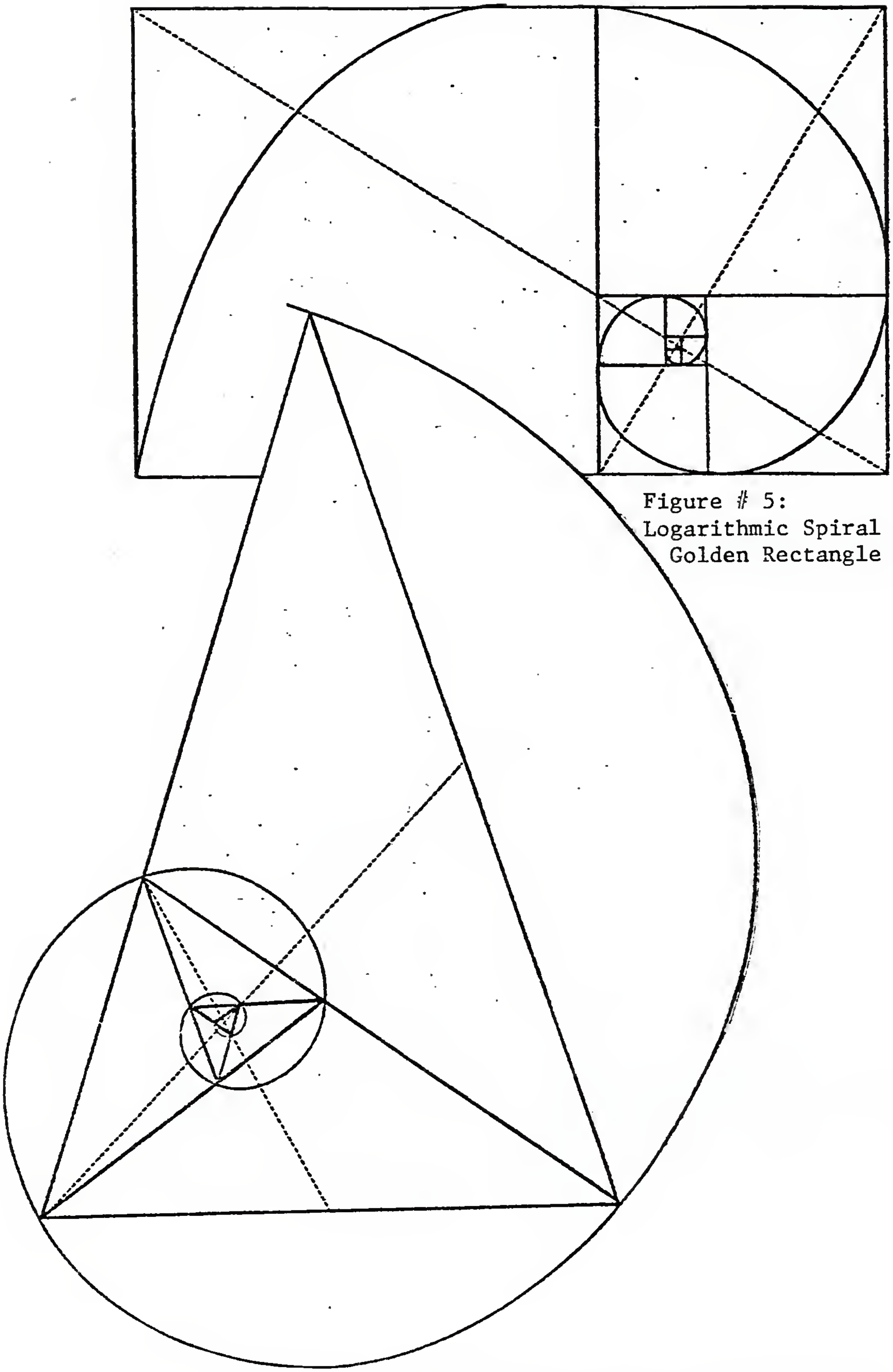


Figure # 5: Logarithmic Spiral & Golden Rectangle

Figure # 4: Logarithmic Spiral & Golden Triangle

> larch, the normal number is eight rows in the one direction and five in the other; while in the American larch we have again three in the one direction and five in the other. It not seldom happens that two arrangements grade into one another on different parts of one and the same cone. Among other cases in which such spiral series are readily visible we have, for instance, the crowded leaves of the stone-crops and mesembryanthemums, and . . . the crowded florets of the composites. Among these we may find plenty of examples in which the numbers of the serial rows are similar to those of the fir-cones; but in some cases, as in the daisy and others of the smaller composites, we shall be able to trace thirteen rows in one direction and twenty-one in the other, or perhaps twenty-one and thirty-four; while in a great big sunflower we may find (in one and the same species) thirty-four and fifty-five, fifty-five and eighty-nine, or even as many as eighty-nine and one hundred and forty-four. On the other hand, in an ordinary "pentamerous" flower, such as a ranunculus, we may be able to trace, in the arrangement of its sepals, petals and stamens, shorter spiral series, three in one direction and two in the other; and the scales on the little cone of a Cypress shew the same numerical simplicity. It will be at once observed that these arrangements manifest themselves in connection with very different things, in the orderly interspacing of single leaves and of entire florets, and among all kinds of leaf-like structures, foliage-leaves, bracts, cone-scales, and the various parts or members of the flower. (Thompson, 1968, vol. 2, pp. 921-922)

Descartes became fascinated with the logarithmic spiral based on the golden section (see Figure # 5, p.153). "The first [modern exoteric] discussions of this spiral occur in letters written by Descartes to Mersenne in 1638" (Hambidge, 1920, p. 146). It should also be noted that three of the propositions in Newton's <u>Principia</u> are based

on the logarithmic spiral. These are Book I, proposition 9,and Book II, propositions 15 and 16.

Kepler was fascinated by the golden section. In his writings he calls it the "sectio divina" and the "proportio divina." He was presumably following the appellation given it by Da Vinci's friend, Fra Luca Pacioli, who in 1509 referred to it as the "divine proportion." Kepler went on to write:

> Geometry has two great treasures, one is the Theorem of Pythagoras, the other the division of a line into extreme and mean ratio; the first we may compare to a measure of gold, the second we may name a precious jewel. (quoted in Hambidge, 1920, p. 153)

Also the Fibonacci "series was well known to Kepler, who discusses and connects it with [the] golden section and growth, in a passage of his 'De nive sexangula', 1611" (Hambidge, 1920, p. 155). Not only did Kepler employ the abductive, or analytic techniques so wonderfully developed by Plato, but he also followed the latter into the significance of the golden section and its approximating whole numbers, the Fibonacci series.

> After discussions of the form of the bees' cells and of the rhombo-dodecahedral form of the seeds of the pomegranite (caused by equalizing pressure) he [Kepler] turns to the structure of flowers whose peculiarities, especially in connection with quincuncial arrangement he looks upon as an emanation of sense of form, and feeling for beauty, from the soul of the plant. He then "unfolds some other reflections" on two regular solids the dodecagon and icosahedron "the former of which is made up entirely of pentagons, the latter of triangles arranged in pentagonal

> form. The structure of these solids in a form so strikingly pentagonal could not come to pass apart from that proportion which geometers to-day pronounce divine." In discussing this divine proportion he arrives at the series of numbers 1, 1, 2, 3, 5, 8, 13, 21 and concludes: "For we will always have as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost. I think that the seminal faculty is developed in a way analogous to this proportion which perpetuates itself, and so in the flower is displayed a pentagonal standard, so to speak. I let pass all other considerations which might be adduced by the most delightful study to establish this truth." (Hambidge, 1920, p. 155)

Goethe discovered the golden section directed logarithmic spiral in the growth of the tails and horns of animals. X-rays of shell growth and horn development (antelopes, wild goats, sheep) provide astounding examples of logarithmic spiral growth based on the golden section.

> . . . in the shells Murex, Fusus Antiquus, Scalaria Pretiosa, Solarium Trochleare, and many fossil Ammonites . . . the specific spiral has the 4th root of Φ as characteristic rectangle or ratio, or quadrantal pulsation, and therefore Φ as radial pulsation. (Ghyka, 1946, p. 94)

Of course the golden section directed logarithmic spiral is evident in the shell of the chambered nautilus (Nautilus pompilius). This is especially true if one is able to observe a radiograph of the shell. It is clear that as the shell grows the chambers increase in size, but the shape remains the same.

The golden section also plays a dominant role in the proportions of the human body. According to Ghyka this fact was recognized by the Greek sculptors

> who liked to put into evidence a parallelism between the proportions of the ideal Temple and of the human body. . . . [Thus] the bones of the fingers form a diminishing series of three terms, a continuous proportion 1, $1/\phi$, $1/\phi^2$ (in which the first, longest term, is equal to the sum of the two following ones), but the most important appearance of the golden section is in the ratio of the total height to the (vertical) height of the navel; this in a well-built body is always ϕ=1.618... or a near approximation like 8/5=1.6 or 13/8=1.625. . . . if one measures this ratio for a great number of male and female bodies, the average ratio obtained will be 1.618. It is probable that the famous canon of Polycletes (of which his "Doryphoros" was supposed to be an example), was based on this dominant role of the golden section in the proportions of the human body; this role was rediscovered about 1850 by Zeysing, who also recognized its importance in the morphology of the animal world in general, in botany, in Greek Architecture (Parthenon) and in music. The American Jay Hambidge . . . guided by a line in Plato's Theaetetus about "dynamic symmetry," established carefully the proportions and probable designs not only of many Greek temples and of the best Greek vases in the Boston Museum, but also measured hundreds of skeletons, including "ideal" specimens from American medical colleges, and confirmed Zeysing's results. . . . (Ghyka, 1946, pp. 16-17)

It is not at all surprising that Hambidge found his clue to "dynamic symmetry" in the Theaetetus. My own view is that Plato probably saw the interplay and infusion of the golden section and Fibonacci series in nature. The ideal numbers, in terms of ratios and proportions, were infused into the Cosmos by the world-soul for Plato. But he was not referring to some mere abstract mathematics. Rather, Plato "saw" the concrete realization of these numbers in the sensible world. He saw this interplay of

proportion in the doctrine of the five regular solids, which his Pythagorean mentors had transmitted to him. Next we will consider these regular solids, the only ones perfectly inscribable in a sphere, to see if they really do embody these proportions.

The Regular Solids

It is recorded that over the doors of his Academy, Plato wrote: "let noone unversed in geometry come under my roof [madeis ageometratos eisito mou tan stegan]" (Thomas, 1957, vol. 1, p. 387). The Pythagorean mathematical disciplines were the essence of his doctrine. The care with which geometry and the other dianoetic disciplines were treated in the Academy is unequalled in our time. Long before Saccheri, Lobachevsky, and Riemann began to consider the nature of non-Euclidean geometries, the Academy in Plato and Aristotle's time had extensively considered them. But the hypothesis of the right angle was adopted over the acute and obtuse angle hypotheses (see Toth, 1969, pp. 87-98).

Likewise the nature of the five Platonic solids (see Figure # 6, p. 159), was thoroughly investigated. In fact the later work, Euclid's Elements, was viewed as designed to show the construction of the five solids out of points, lines, and planes.

> A regular solid is one having all its faces equal polygons and all its solid angles equal. The term is usually restricted to those regular solids in which the centre is

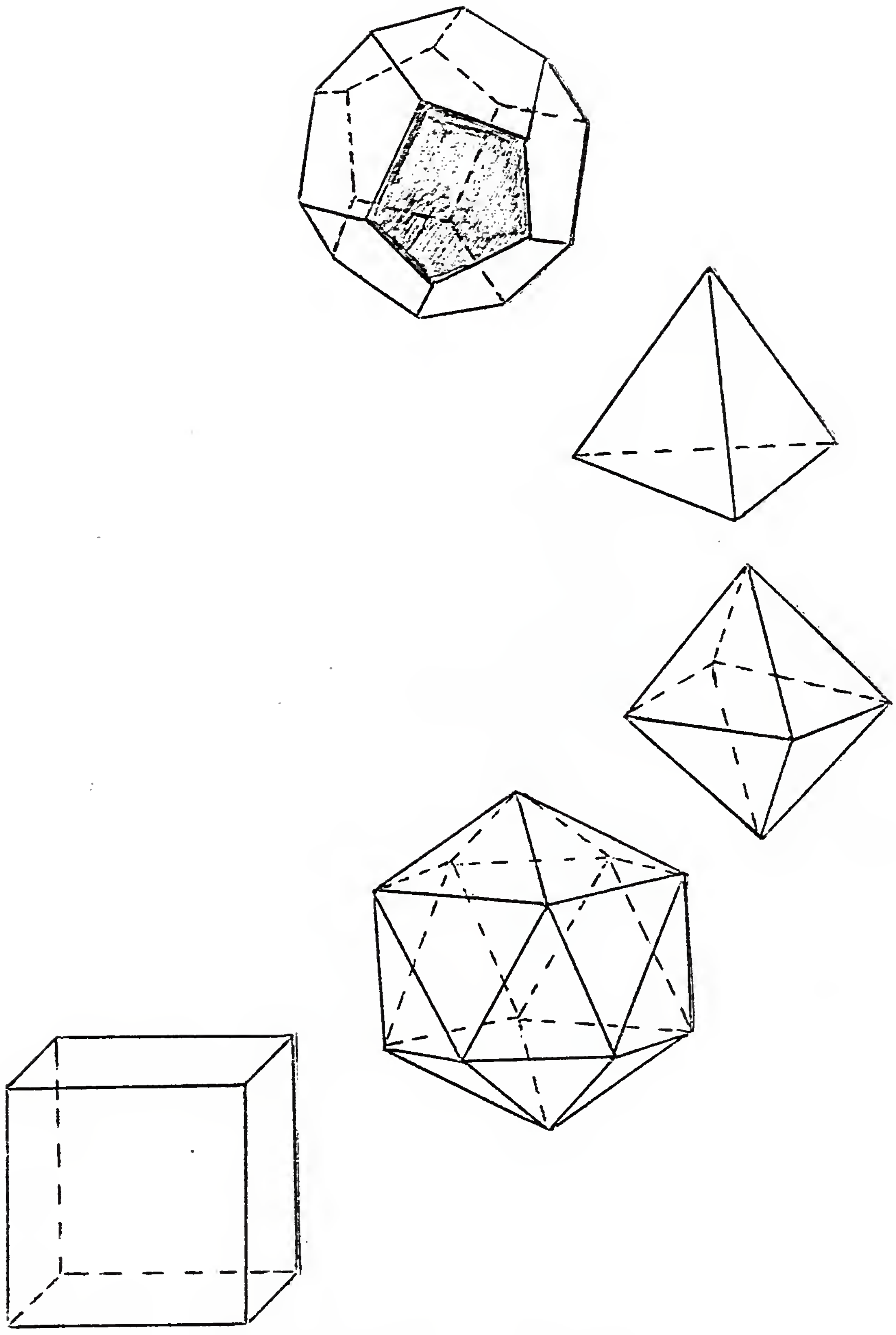


Figure # 6: Five Regular Solids

> singly enclosed. They are five, and only five, such figures--the pyramid, cube octahedron, dodecahedron and icosahedron. They can all be inscribed in a sphere. Owing to the use made of them in Plato's *Timaeus* for the construction of the universe they were often called by the Greeks the cosmic or Platonic figures. (Thomas, 1957, vol. 1, p. 216, fn.a)

The tradition is that Plato received his ideas regarding the regular solids from the Pythagoreans. It is not clear whether his source was to have been Philolaus, Archytas, or Alcmaeon.

> Proclus attributes the construction of the cosmic figures to Pythagoras, but Suidas says Theaetetus was the first to write on them. The theoretical construction of the regular solids and the calculation of their sides in terms of the circumscribed sphere occupies Book XIII of Euclid's *Elements*. (Thomas, 1957, vol. 1, p. 216, fn.a)

It is recorded by Aetius, presumably upon the authority of Theophrastus, that,

> Pythagoras, seeing that there are five solid figures, which are also called the mathematical figures, says that the earth arose from the cube, fire from the pyramid, air from the octahedron, water from the icosahedron, and the sphere of the universe from the dodecahedron. (Thomas, 1957, vol. 1, p. 217)

This is precisely the characterization given to the elements by Plato. But Plato does something surprising in the Timaeus in the construction of the solids (i.e., *Timaeus* 53c-55c). He constructs them out of two kinds of triangles. Why is there not a further reduction from triangular planes, to lines, and from lines to ratios of numbers? He appears to stop prematurely.[1]

> God now fashioned them by form and number. . . . God made them as far as possible the fairest and best, out of things which were not fair and good. . . . fire and earth and water and air are bodies. And every sort of body possesses volume, and every volume must necessarily be bounded by surfaces, and every rectilinear surface is composed of triangles, and all triangles are originally of two kinds, both of which are made up of one right and two acute angles. . . . These, then, proceeding by a combination of probability with demonstration, we assume to be the original elements of fire and the other bodies, but the principles which are prior to these God only knows, and he of men who is the friend of God. And next we have to determine what are the four most beautiful bodies which could be formed, unlike one another, yet in some instances capable of resolution into one another, for having discovered thus much, we shall know the true origin of earth and fire and of the proportionate and intermediate elements. Wherefore we must endeavor to construct the four forms of bodies which excel in beauty, and secure the right to say that we have sufficiently apprehended their nature. (*Timaeus* 53b-e)

This is a difficult and at times cryptic passage. It should be noted that it is "put into the mouth of Timaeus of Locri, a Pythagorean leader, and in it Plato is generally held to be reproducing Pythagorean ideas" (Thomas, 1957, vol. 1, p.218, fn.a). There is the hint that he who is a friend of God, and therefore one on the upward path of analytic ascent near to the Forms, may know the principles prior to the triangular components of the solids.

But first, what can be said about Plato's primitive triangles? The first triangle is the isosceles right-angled triangle, or half-square, whose sides are in

the ratio $1:1:\sqrt{2}$, (see Figure # 7, p. 163). Out of this triangle is constructed the cube or element of earth. Out of the many right-angled scalene triangles Plato chooses as his second primitive, the one which is a half-equilateral triangle. He calls it the most bequtiful of the scalene triangles. Its sides are in the ratio of $1:\sqrt{3}:2$ (see Figure # 8 , p. 163). This latter triangle is to be used in the construction of the elements of fire (tetrahedron), air (octahedron), and water (icosahedron).

Plato typifies these triangles as the most beautiful, making it evident that he is equally concerned with aesthetics here. As Vlastos notes:

> It would be hard to think of a physical theory in which aesthetic considerations have been more prominent: [Vlastos quoting Plato passages] "Now the question to be considered is this: What are the most beautiful (kallista) bodies that can be constructed [Timaeus] (53d7-e1). . . ." "If we can hit upon the answer to this question we have the truth concerning the generation of earth and fire (e3-4). . . . " "For we will concede to no one that there are visible bodies more beautiful (kallio) than these (e4-5)." "This then we must bestir ourselves to do: construct four kinds of body of surpassing beauty (diaferonta kallei) and declare that we have reached a sufficient grasp of their nature (e6-8). . . ." "Of the infinitely many [scalene triangles] we must prefer the most beautiful (to kalliston) (54a2-3)." (Vlastos, 1975, p. 93, fn.41)

We know from Speusippus that Plato's isosceles right-angled triangle represented the dyad and even numbers. Plato's scalene right-angled triangle (the half-equilateral) represented the triad and odd numbers. But there was also, according to Speusippus, the

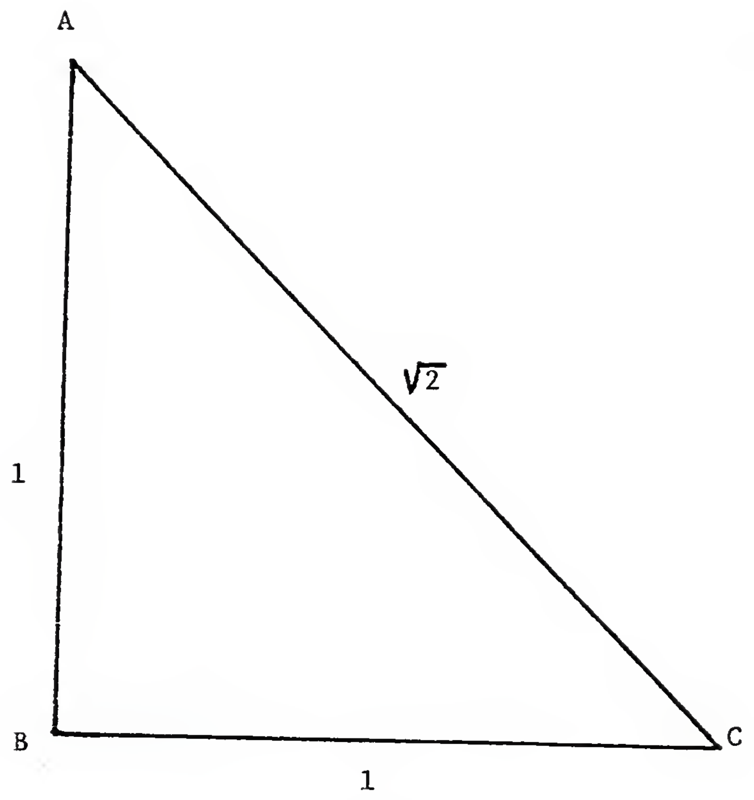


Figure # 7: 1:1:$\sqrt{2}$ Right-angled Isosceles Triangle

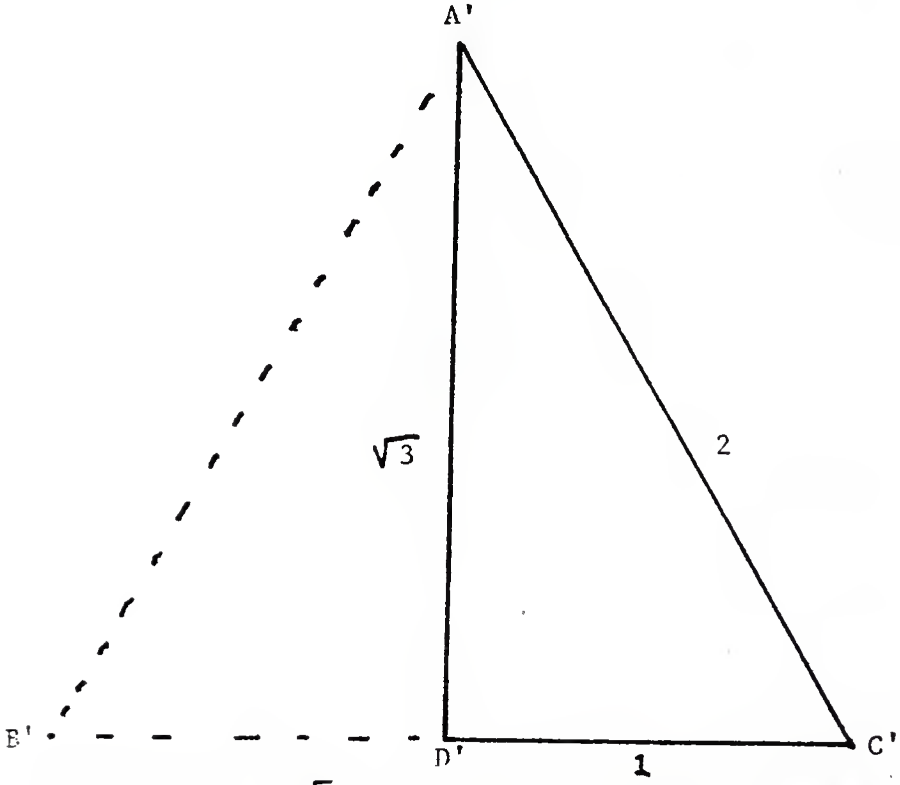


Figure # 8: 1:$\sqrt{3}$:2 Right-angled Scalene Triangle

equilateral triangle which represented the monad (see Figure # 9, p. 165). This triangle occurs in the Timaeus. But rather than being a primitive there, it is constructed out of the scalene right-angled triangles. It is then employed in the construction of the elements. Referring to triangular plane figures from the point of view of number in his book, On Pythagorean Numbers, Speusippus says:

> For the first triangle is the equilateral which has one side and angle; I say one because they are equal; for the equal is always indivisible and uniform. The second triangle is the half-square; for with one difference in the sides and angles it corresponds to the dyad. The third is the half-triangle which is half of the equilateral triangle; for being completely unequal in every respect, its elements number three. In the case of solids, you find this property also; but going up to four, so that the decad is reached in this way also. (Thomas, 1957, vol. 1, pp. 80-83)

It is interesting that here the equilateral triangle is the first primitive, whereas, in the Timaeus the equilateral triangle is built up out of the half-equilaterals (i.e., out of the scalene right-angled primitives). Nevertheless, this work of Speusippus' On Pythagorean Numbers tends to reinforce the argument for the Pythagorean origins of Plato's doctrine in the Timaeus. But note further that this Pythagorean "awareness" of Plato dates back to the earliest dialogues. Reference to these primitive triangles is made as early as the Euthyphro (note Figure # 10, p. 166).

> Suppose . . . you had asked me what part of number is the even, and which the even number is. I would have said that it is the one

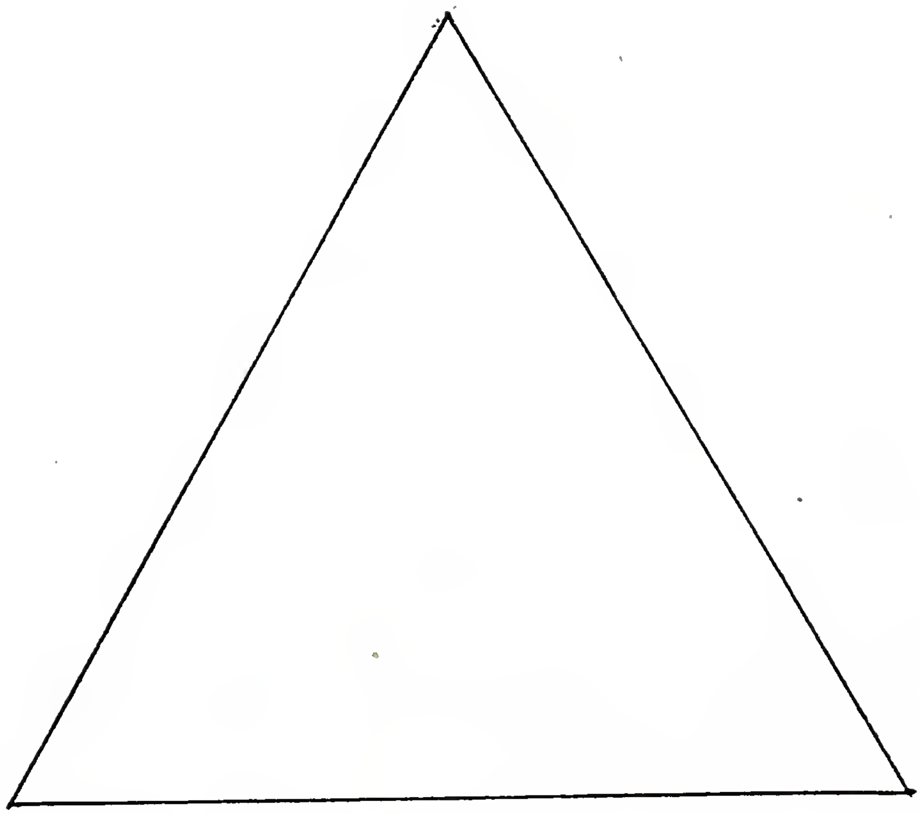


Figure # 9: Monadic Equlateral Triangle

Arnim	Lutoslawski	Raeder	Ritter	Wilamowitz
	Apol.	Apol.	Hipp. Mi.	Ion
Ion		Ion		Hipp. Mi.
Prot.		Hipp. Mi.	Lach.	Prot.
Lach.	Euthyph.	Lach.	Prot.	Apol.
Rep. 1	Crito	Charm.	Charm.	Crito
Lysis	Charm.	Crito	Euthyph.	Lach.
Charm.		Hipp. Ma.	Apol.	
Euthyph.	Lach.	Prot.	Crito	Lysis
Euthyd.	Prot.	Gorg.	Gorg.	Charm.
Gorg.		Menex.	Hipp. Ma.	Euthyph.
Meno	Meno	Euthyph.	Euthyd.	Gorg.
Hipp. Mi.	Euthyd.	Meno	Crat.	Menex.
Crat.	Gorg.	Euthyd.	Meno	Meno
Symp.	Rep. 1	Crat.	Menex.	Crat.
Hipp. Ma.	Crat.	Lysis	Lysis	Euthyd.
Phaedo	Symp.	Symp.	Symp.	Phaedo
Crito	Phaedo	Phaedo	Phaedo	Symp.
Rep. 2-10	Rep. 2-10	Rep.	Rep.	Rep.
Theaet.	Phaedr.	Phaedr.	Phaedr.	Phaedr.
Parm.	Theaet.	Theaet.	Theaet.	Parm.
Phaedr.	Parm.	Parm.	Parm.	Theaet.
Soph.	Soph.	Soph.	Soph.	Soph.
Pol.	Pol.	Pol.	Pol.	Pol.
Phil.	Phil.	Phil.	Tim.	Tim.
	Tim.	Tim.	Critias	Critias
	Critias	Critias	Phil.	Phil.
Laws	Laws	Laws	Laws	Laws
		Epin.		

Figure # 10: Stylometric Datings of Plato's Dialogues
(after Ross, 1951, p. 2)

> that corresponds to the isosceles, and not the scalene. (Euthyphro 12d)

There appear, however, to be difficulties with the doctrine in the Timaeus. The beauty of the Pythagorean number theory is manifest. But why is the cube presumably not transformable into the other elements? And why is the fifth element which is to stand as the foundation of the Cosmos, the dodecahedron, not transformable into any of the other four elements?

Vlastos brings out this typical criticism when he says:

> I am not suggesting that the aesthetics of the Platonic theory are flawless. The exclusion of earth from the combinatorial scheme (necessitated by the noninterchangeability of its cubical faces with the triangular faces of the solids assigned to fire [tetrahedron], air [octahedron], and water [icosahedron]) is awkward. Worse yet is the role of the fifth regular solid, the dodecahedron, whose properties would also shut it out of the combinatorial cycle. The hasty reference to it (55c) suggests embarrassed uncertainty. What could he mean by saying that "the god used it for the whole?" The commentators have taken him to mean that the Demiurge made the shape of the Universe a dodecahedron; this unhappily contradicts the firm and unambiguous doctrine of 33b (reaffirmed in 43d and 62d) that the shape of the Universe is spherical. (Vlastos, 1975, p. 94, fn.43)

Irrespective of what Vlastos claims, it is evident that Plato intended the dodecahedron to be of fundamental importance in the structure of the Cosmos. And furthermore, as I will show, the terse reference at Timaeus 55c is not due to "embarrassed uncertainty," but to reticence and midwifery on the part of Plato. Plato is reticent because he is discussing what are probably some of

the most cherished of Pythagorean doctrines. He is playing midwife because while again suggesting that the friend of deity can penetrate deeper, he at the same time presents the problem: an analysis into primitives which has not been carried far enough. I will answer Vlastos' charges directly, and return to this question of primitive triangles shortly.

The regular bodies are avowed to be the most beautiful. And then we are told that we must "secure the right to say that we have sufficiently apprehended their nature" (Timaeus 53e). But why does Plato stop short in the analysis to the most beautiful component? The answer lies in the fact that Plato must use some form of blind or cover. The deepest Pythagorean discoveries were not to be openly revealed. And as Plato has pointed out in the Phaedrus and the 7th Letter, he is opposed to openly exposing one's doctrine in writing. And yet Plato provides sufficient hints, as he continues to be obstetric in the Timaeus, as he has been in other dialogues.

We must also inquire as to why the fifth figure, or dodecahedron, was singled out as the element of the Cosmos? "There was yet a fifth combination which God used in the delineation of the universe with figures of animals" (Timaeus 55c). Was there something special about the dodecahedron? We do have one story preserved by Iamblichus in his On the Pythagorean Life.

> It is related of Hippasus that he was a Pythagorean, and that, owing to his being the

> first to publish and describe the sphere from the twelve pentagons, he perished at sea for his impiety, but he received credit for the discovery, though really it all belonged to HIM (for in this way they refer to Pythagoras, and they do not call him by his name). (Thomas, 1957, vol. 1, pp. 223-225)

We do know that a dodecahedron of course has 12 pentagonal faces. The pentagram which is created by connecting opposing vertices of the pentagon was the sacred symbol of the Pythagorean brotherhood.

> The triple interlaced triangle, the pentagram, which they (the Pythagoreans) used as a password among members of the same school, was called by them Health. (Lucian in Thomas, 1957, vol. 1, p. 225)

Are there any other clues? Yes, there are some very relevant clues. Returning to Heath's comments upon the ancient methods of analysis and synthesis, we may begin to be able to begin tieing together the notions of analysis and that of the golden section. This begins to emerge when we consider Book XIII of Euclid's Elements.

> It will be seen from the note on Eucl. XIII. 1 that the MSS. of the Elements contain definitions of Analysis and Synthesis followed by alternative proofs of XIII 1-5 after that method. The definitions and alternative proofs are interpolated because of the possibility that they represent an ancient method of dealing with these propositions, anterior to Euclid. The propositions give properties of a line cut "in extreme and mean ratio," and they are preliminary to the construction and comparison of the five regular solids. (Heath, 1956, vol. 1, p. 137)

My contention, simply put, is that this lies at the center of what was going on in Plato's Academy. The line

cut in extreme and mean ratio was central to these endeavours. In fact, if one takes the fifth solid, the dodecahedron, it is discovered that its analysis into plane surfaces, or the second dimension, yields the pentagons. The further analysis into one dimension produces the relation of a line cut in mean and extreme ratio. And finally, reduced to the level of number we have the relation of Φ to 1.

> Now Pappus, in the section of his Collection dealing with [the construction and comparison of the regular solids] says that he will give the comparisons between the five figures, the pyramid, cube, octahedron, dodecahedron and icosahedron, which have equal surfaces, "not by means of the so-called analytical inquiry, by which some of the ancients worked out the proofs, but by the synthetical method." The conjecture of Bretschneider that the matter interpolated in Eucl. XIII. is a survival of investigations due to Eudoxus has at first sight much to commend it. In the first place, we are told by Proclus that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section, and employed in them the method of analysis." It is obvious that "the section" was some particular section which by the time of Plato had assumed great importance; and the one section of which this can safely be said is that which was called the "golden section," namely, the division of a straight line in extreme and mean ratio which appears in Eucl. II.11 and is therefore most probably Pythagorean. (Heath, 1956, vol. 1, P. 137)

Here is the anomalous evidence that the Pythagorean Platonists, namely Plato, Eudoxus, and perhaps Theaetetus, Speusippus, and Xenocrates, were busying themselves with a consideration of the golden section through a dialectical technique of analysis, which was essentially a series of reductions. This golden section was the mathematical (par

excellence) of the incommensurables. It not only is essential to an understanding of the relationships of the regular solids, but is also fundamental to a theory of proportions based upon incommensurables. Just as the regular solids and the world soul were shown to embody the geometrical proportions of the whole numbers (e.g., 2:4::4:8), in the same way that the Sun, Divided Line, and Cave similes were related one to another, so the regular solids embody in their relations the ratio of the golden section, just as the Divided Line embodied this geometrical proportion of incommensurables.

> As Cantor points out, Eudoxus was the founder of the theory of proportions in the form in which we find it in Euclid V., VI., and it was no doubt through meeting, in the course of his investigations, with proportions not expressible by whole numbers that he came to realise the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes. The "golden section" would furnish such a case. And it is even mentioned by Proclus in this connexion. He is explaining that it is only in arithmetic that all quantities bear "rational" ratios (ratos logos) to one another, while in geometry there are "irrational" ones (arratos) as well. "Theorems about sections like those in Euclid's second Book are common to both [arithmetic and geometry] except that in which the straight line is cut in extreme and mean ratio. (Heath, 1956, vol. 1, p. 137)

Now it appears that Theaetetus was also intimately involved with this consideration of the golden section and its relation to the five regular solids.

> It is true that the author of the scholium No. 1 to Eucl. XIII. says that the Book is about "the five so-called Platonic figures, which however do not belong to Plato, three

> of the aforesaid five figures being due to the Pythagoreans, namely the cube, the pyramid and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus." This statement (taken probably from Geminus) may perhaps rest on the fact that Theaetetus was the first to write at any length about the two last-mentioned solids. We are told indeed by Suidas (s.v. Theaitatos) that Theaetetus "first wrote on the 'five solids' as they are called." This no doubt means that Theaetetus was the first to write a complete and systematic treatise on all the regular solids; it does not exclude the possibility that Hippasus or others had already written on the dodecahedron. The fact that Theaetetus wrote upon the regular solids agrees well with the evidence which we possess of his contributions to the theory of irrationals, the connexion between which and the investigation of the regular solids is seen in Euclid's Book XIII. (Heath, 1956, vol. 3, p. 438)

Santillana is one of the few modern scholars who has sincerely tried to penetrate the Platonic mystery. He argues as follows:

> . . . that world [the Cosmos] is a dodecahedron. This is what the sphere of twelve pieces [in Phaedo 107d-115a] stands for: there is the same simile in the Timaeus (55c), and then it is said further that the Demiurge had the twelve faces docorated with figures (diazographon) which certainly stand for the signs of the zodiac. A.E. Taylor insisted rather prosily that one cannot suppose the zodiacal band uniformly distributed on a spherical surface, and suggested that Plato (and Plutarch after him) had a dodecagon in mind and they did not know what they were talking about. This is an unsafe way of dealing with Plato, and Professor Taylor's suffisance soon led him to grief. Yet Plutarch had warned him: the dodecahedron "seems to resemble both the Zodiac and the year." (Santillana and von Dechend, 1969, p. 187)

Santillana has in mind a passage where Plutarch relates the pentagons of the dodecahedron into triangles. Plutarch asks:

> . . . is their opinion true who think that he ascribed a dodecahedron to the globe, when he says that God made use of its bases and the obtuseness of its angles, avoiding all rectitude, it is flexible, and by circumtension, like globes made of twelve skins, it becomes circular and comprehensive. For it has twenty solid angles, each of which is contained by three obtuse planes, and each of these contains one and the fifth part of a right angle. Now it is made up of twelve equilateral and equangular quinquangles (or pentagons), each of which consists of thirty of the first scalene triangles. Therefore it seems to resemble both the Zodiac and the year, it being divided into the same number of parts as these. (Plutarch in Santillana and von Dechend, 1969, p. 187)

It is in this way that Santillana sees the golden section emerge from it. Santillana remarks:

> In other words, it is stereometrically the number 12, also the number 30, the number 360 ("the elements which are produced when each pentagon is divided into 5 isosceles triangles and each of the latter into 6 scalene triangles") -- the golden section itself. This is what it means to think like a Pythagorean. (Santillana and von Dechend, 1969, p. 187)

But has Santillana really discovered how the golden section is involved? When Plutarch says above that each pentagon may be divided into "thirty of the first scalene triangles," it is a blind. They are right-angled scalene triangles, but they are not Plato's primitive right-angled scalene triangles. Furthermore, when Santillana says above, that to arrive at this number, the pentagon is first

"divided into 5 isosceles triangles," it is again misleading. Yes, they are isosceles triangles, but they are not Plato's primitive right-angled isosceles triangles. However, by examination of these geometrical reductions, we may get a better grasp of what actually was primitive for Plato.

Looking at Figure # 11, p. 175, we have a pentagon ABCDE. If we then connect each of the corners to the center F, we have the pentagon divided into five isosceles triangles (see Figure #12, p. 176). If we then bisect each isosceles triangle by drawing a line, for example, from A through F projecting it until it meets line CD at N, and doing the same for points B, C, D, and E, we have ten scalene right-angled triangles (see Figure # 13, p. 177). However, they are not Plato's primitive scalene triangles. If we next proceed around the pentagon, first drawing a line from A to C, and then from A to D, then from B to D, and then from B to E, continuing with points C, D, E, in like manner, we arrive at Figure # 14, p. 178. It will now be noticed that each of the five isosceles triangles is composed of six scalene triangles. This results in thirty scalene triangles within the pentagon. However, ten of these thirty triangles are different from the others.

I will try to illustrate the point slightly differently. Beginning with our bare regular pentagon ABCDE (see Figure # 11, p. 175), let us draw lines AC and CE (Figure # 15, p. 179). The resulting triangle is termed

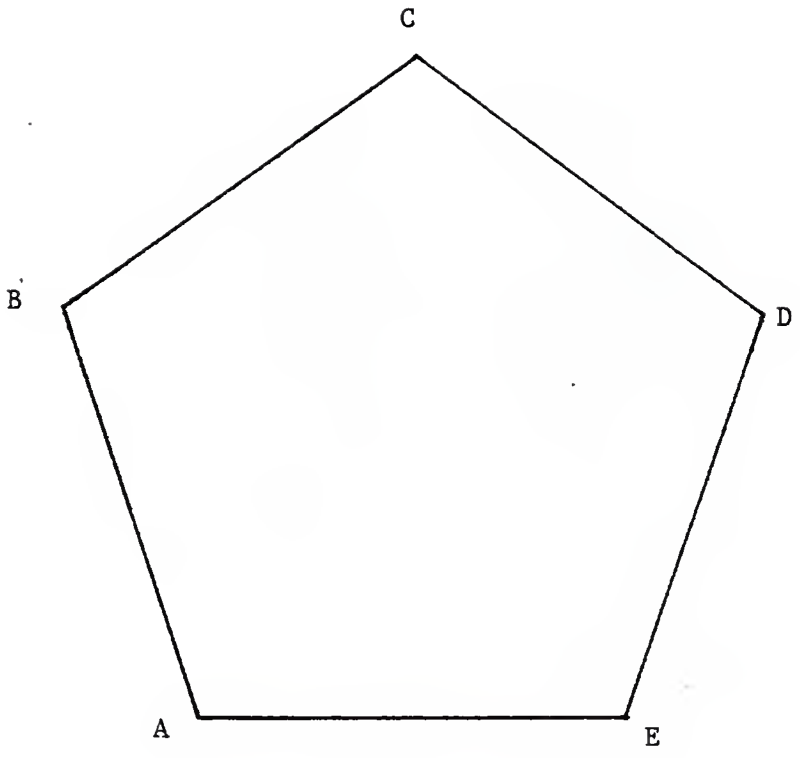


Figure # 11: Pentagon

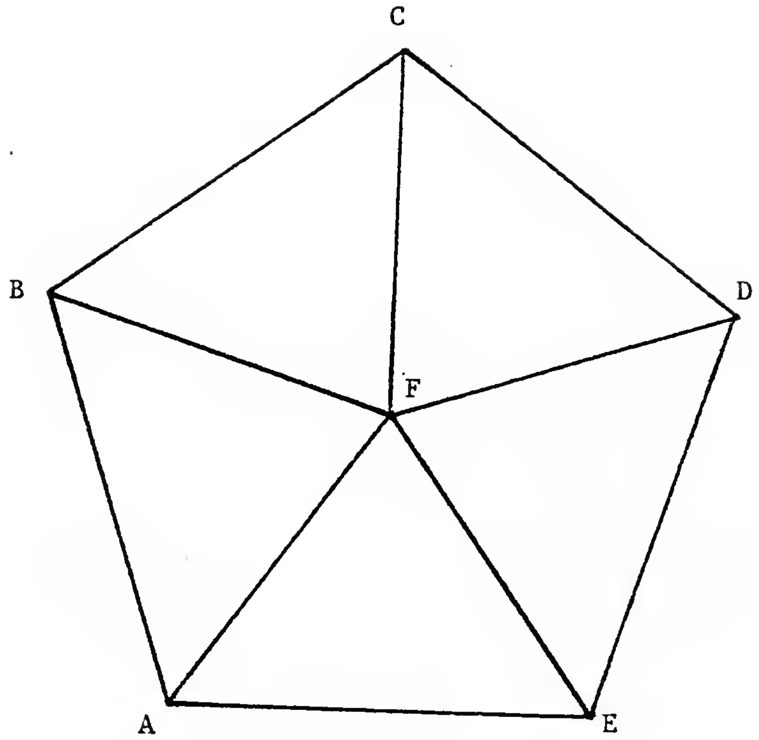


Figure # 12: Pentagon & Isosceles Triangle

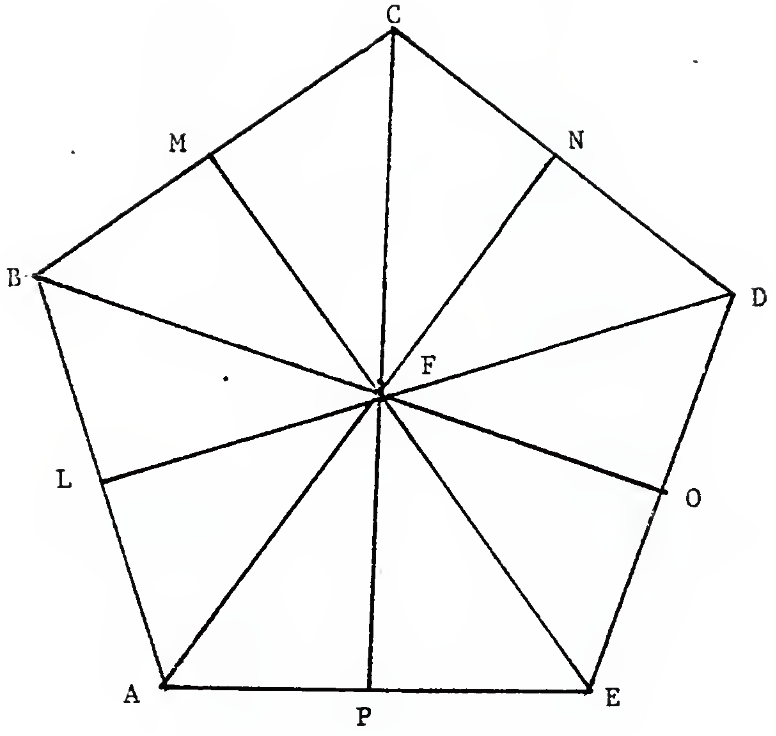


Figure # 13: Pentagon & 10 Scalene Triangles

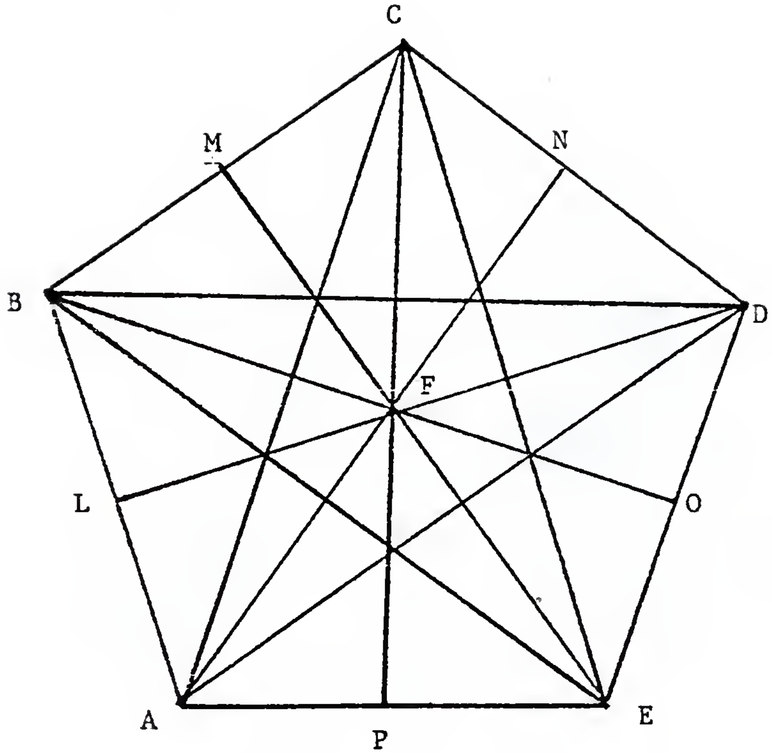


Figure # 14: Pentagona & 30 Scalene Triangles

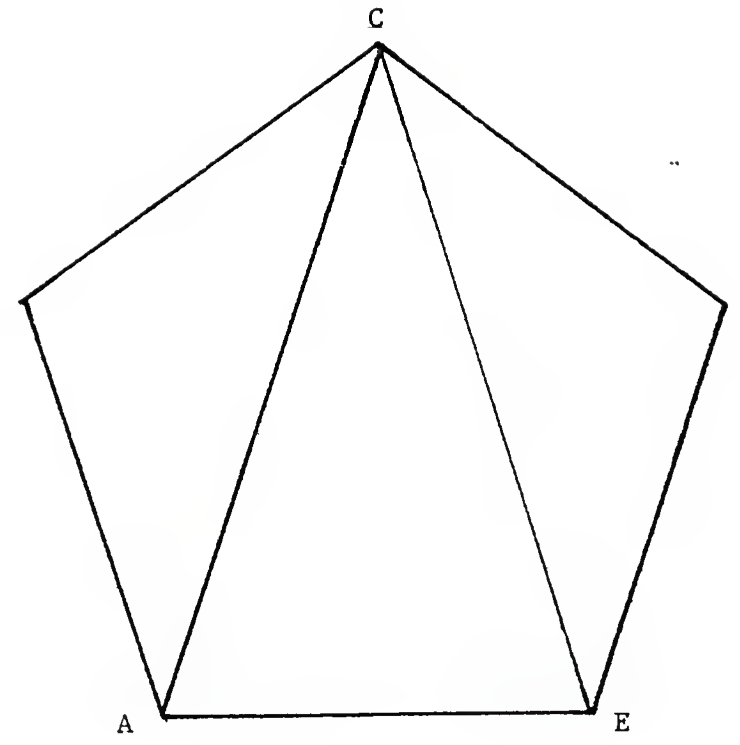


Figure # 15: Pentagon & Pentalpha

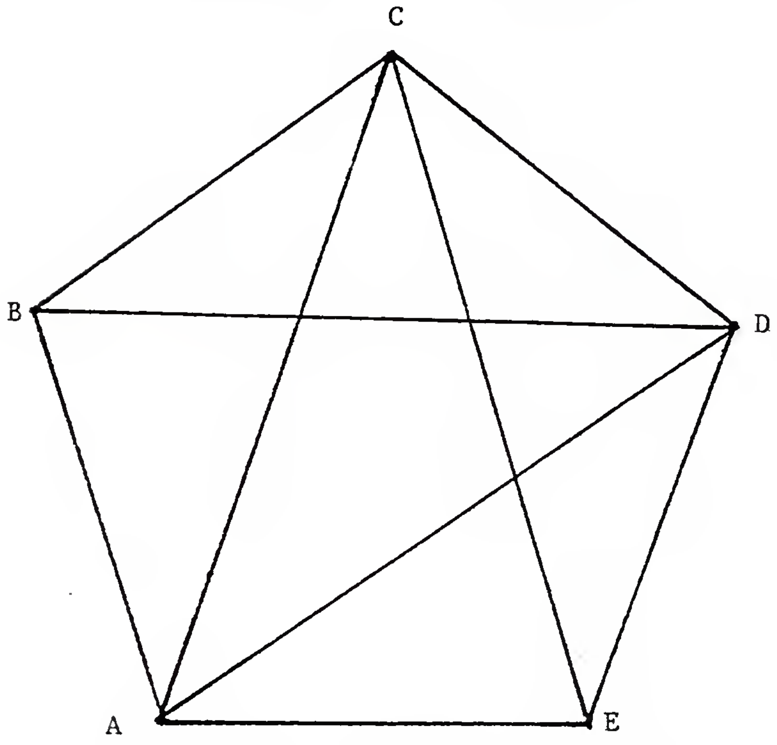


Figure # 16: Pentagon & Two Pentalphas

the pentalpha or golden triangle. If line AC = ϕ, then line AE = 1. If we then draw lines BD and AD, we find two intersecting pentalphas (Figure # 16, p. 179). If we then draw line BE we discover the figure of the pentagram (Figure # 17, p. 181). In effect we have drawn three intersecting pentalphas. The resulting pentagram inscribed in the pentagon can also easily be viewed as five intersecting pentalphas. Next locating the center point F, let us draw lines AF and EF. Then we will draw a line from F perpendicular to AE meeting AE in J. J will be midpoint between A and E (see again Figure # 18, p. 181). Let us then eliminate all internal lines except those within the isosceles triangle AFE (Figure # 19 , p. 182). Let us then extract triangle AFE (with its internal lines intact) from our pentagon (again see Figure # 19 , p. 182).

The first thing to be noticed is that triangles AJI, EJI, FGI, and FHI are all scalene right-angled triangles. However, triangles AGI and EHI are scalene right-angled triangles of a different kind. As already noted, none of these are Plato's primitive scalene triangles. If we then extract the two triangles AGI and EHI, we quickly discover that they are half-pentalphas (Figure # 20 , p. 183). If we then place the two triangles together, back to back, we find that we have a golden triangle or pentalpha (Figure #21, , p. 184). Thus, the half-pentalpha emerges through a division of the pentagram. Because of this we can say that the pentalpha is basic to the construction of the pentagon,

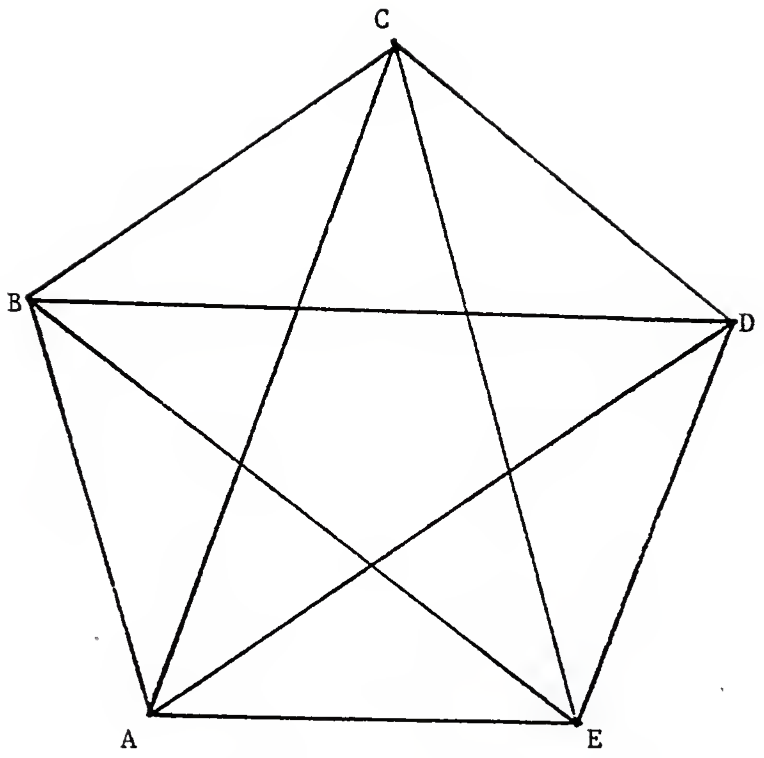


Figure # 17: Pentagon & Pentagram

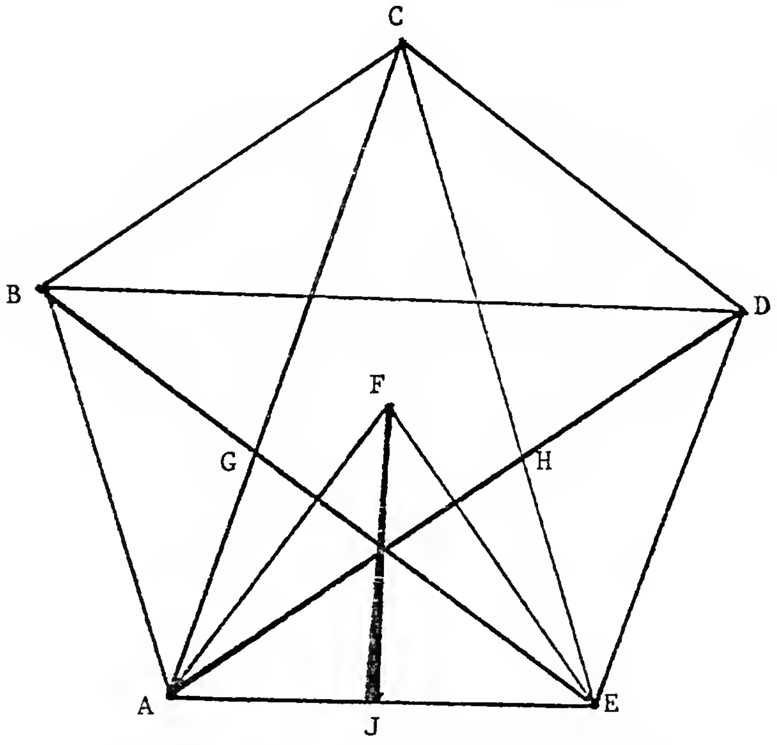


Figure # 18: Pentagonal Bisection

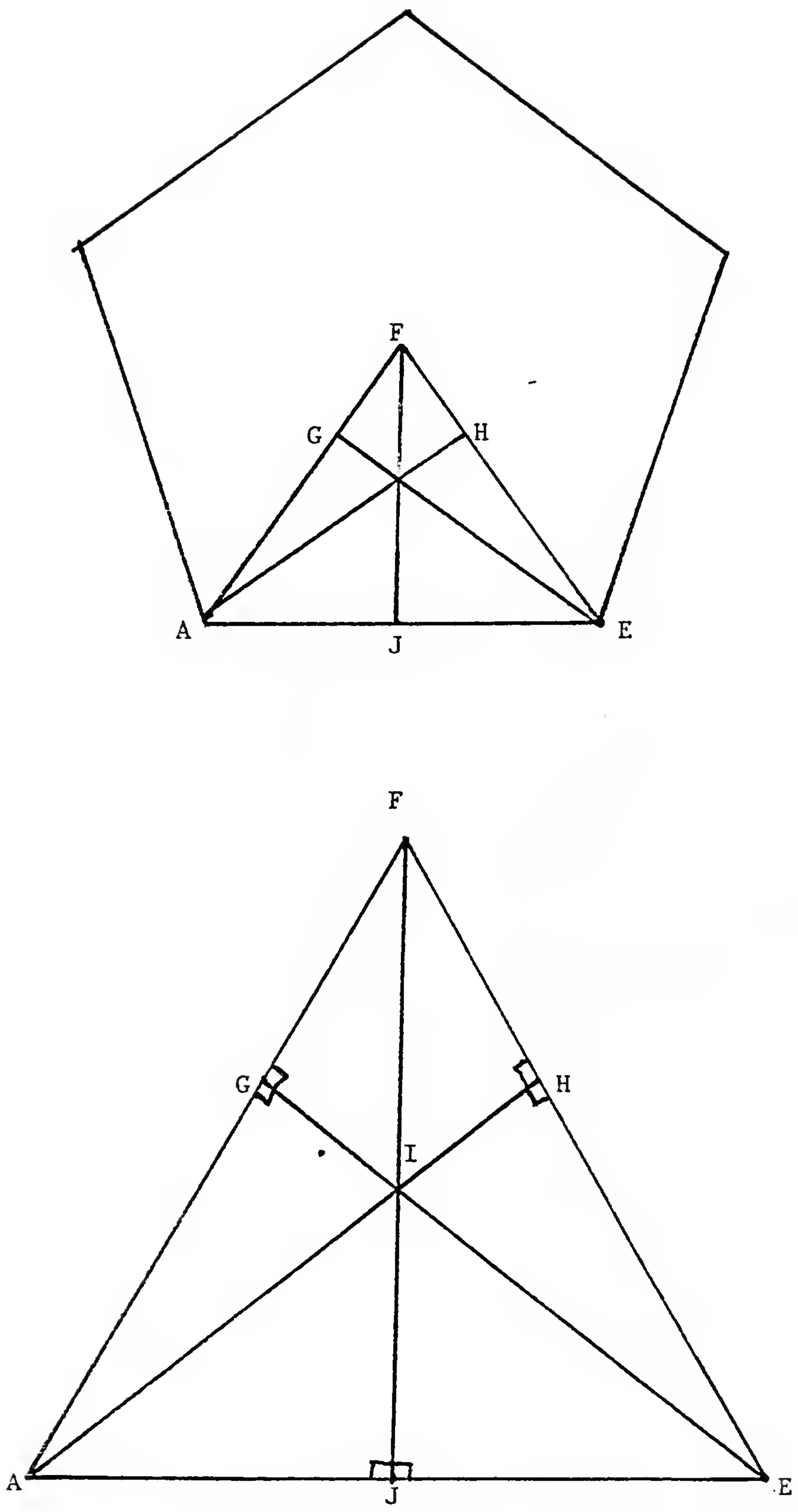


Figure # 19: Pentagon & Isosceles Triangle (top)
(latter extracted at bottom)

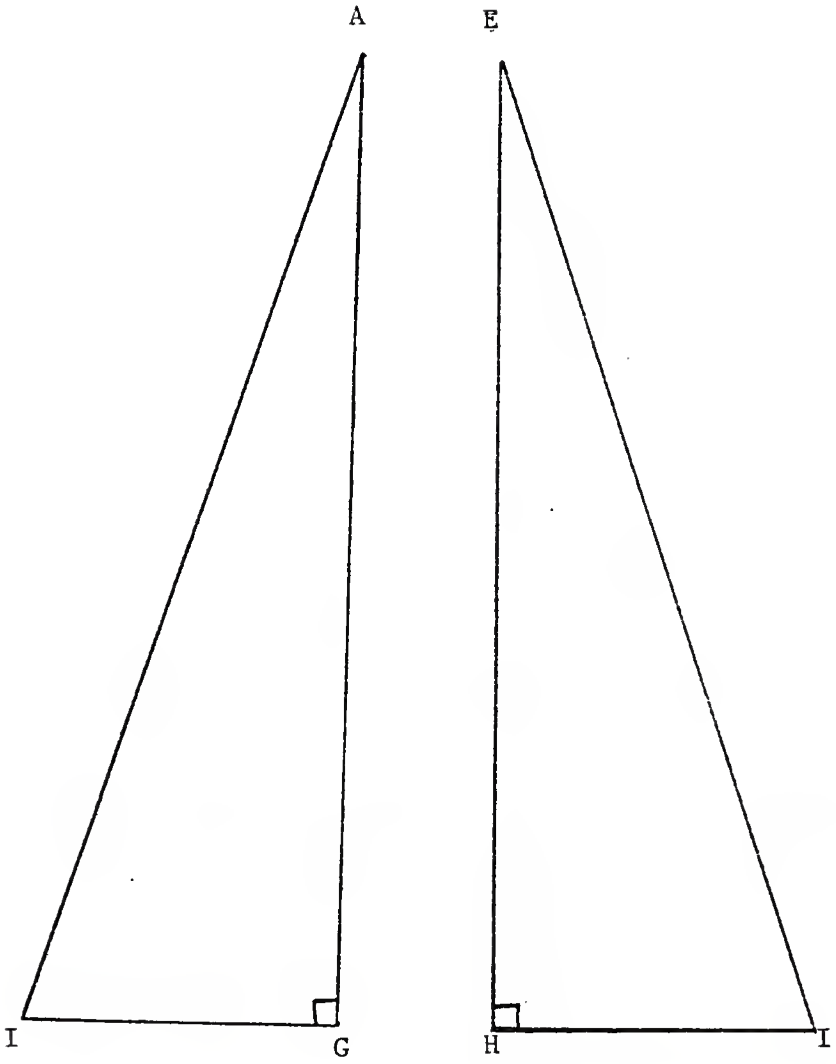


Figure # 20: Two Half-Pentalphas

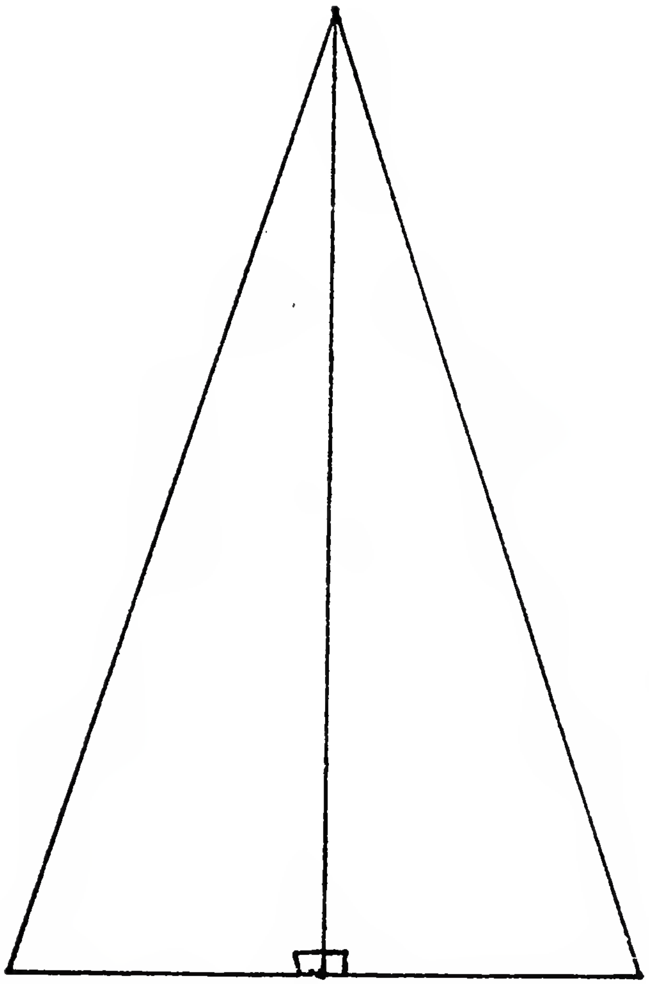


Figure # 21: Pentalpha

and hence, the dodecahedron. If the pentalpha is reduced from a two dimensional plane figure, to its one dimensional depiction in a line, we discover the golden section. This is the fundamental modular of Plato's cosmos.

Next we shall view the same thing but strictly from the perspective of the golden section. Let us begin with a new line AB. Then we will draw BD which is equal to 1/2 AB, perpendicular to AB (see Figure # 22, p. 186). Next we will draw line AD. Then with a compass at center D and with radius DB we will draw an arc cutting DA in E. Next with center A, and radius AE, we will draw an arc cutting AB at C (Figure # 22, p. 186). Line AB is then cut in the golden section at C.

Next let us place the compass at center C and with the radius CA, draw an arc. Then again with radius CA, but with the compass at center B, we draw another arc which cuts the earlier arc at F (see Figure # 23, p. 186). We then draw the lines AF, CF, and BF. We then have constructed the pentalpha BCF. Next by bisecting angles BAF and CFB, we are able to locate the point of intersection I (see Figure # 24, p. 187). Then with the compass at center I and radius IB, we construct the circle BCF (Figure # 25, p. 188). The circle also intersects at points C and J, as well as B and F. Then we extend a line through AI projected to meet the circle BCF in K (again see Figure # 25, p. 188). Next we draw lines BC, CJ, JF, FK, and KB. This gives us pentagon BCJFK inscribed in circle

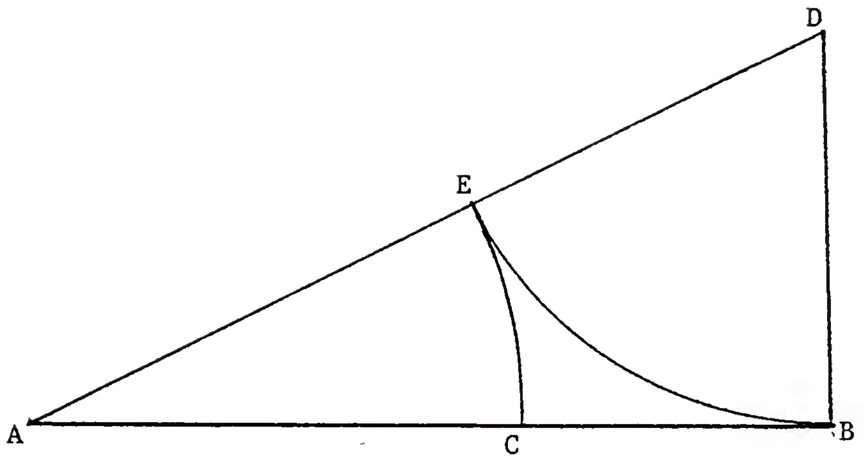


Figure # 22: Golden Cut

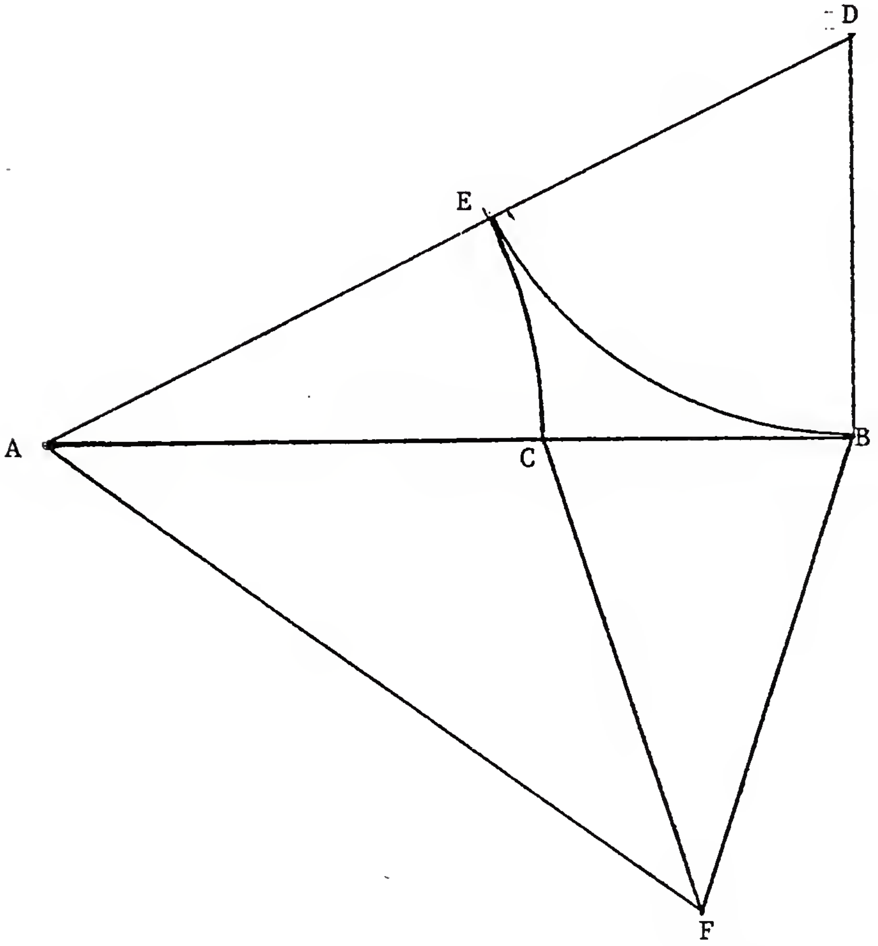


Figure # 23: Golden Cut & Pentalpha

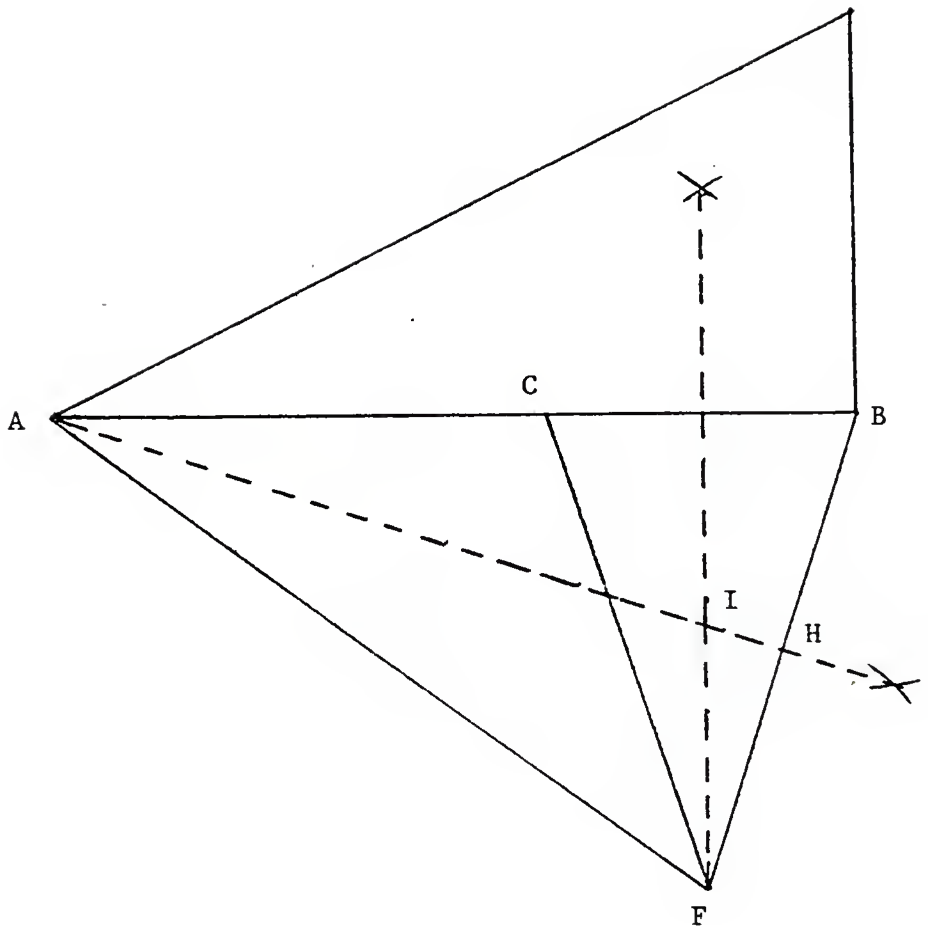


Figure # 24: Pentalpha Bisection

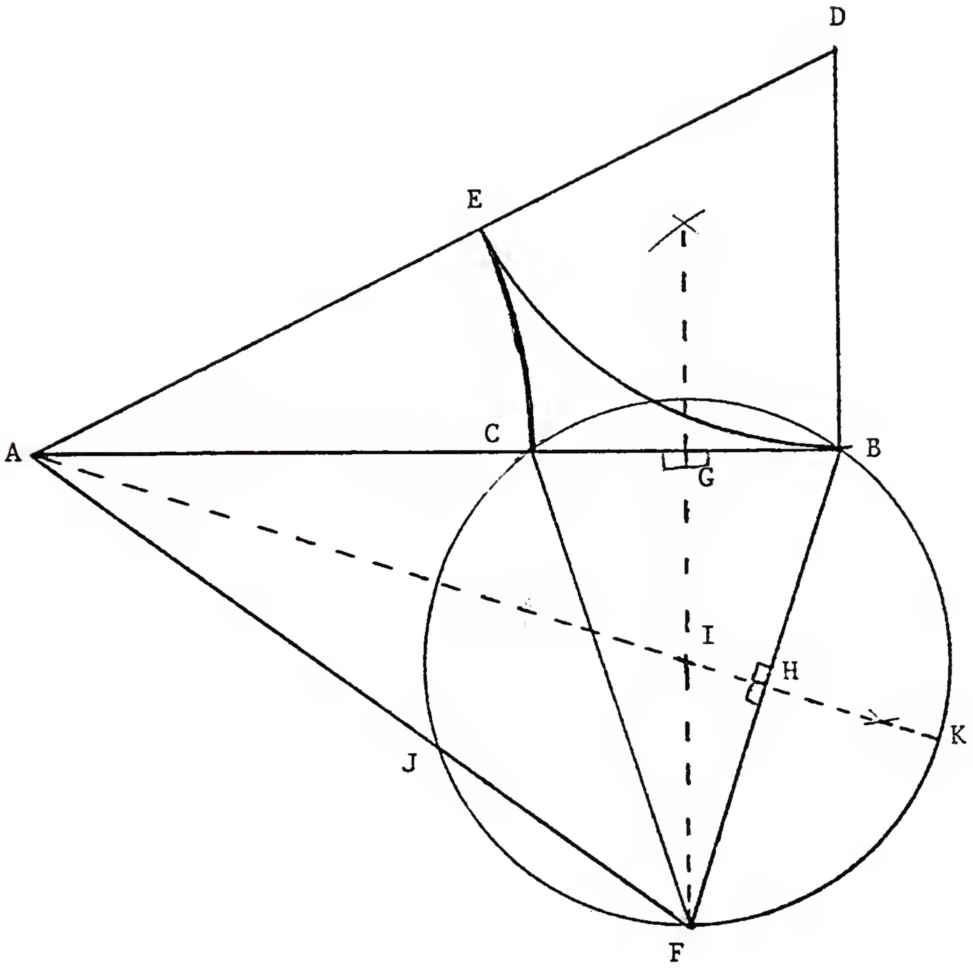


Figure # 25: Circle & Pentalpha

BCF (Figure # 26, p. 190). It would be correct to say at this point we are half-way towards inscribing a dodecahedron in a sphere.

But now let us rotate the entire figure 180°. Point F will now be uppermost (Figure # 27, p. 191). Now let us remove everything except the circle, pentagon, and pentalpha. Then we will drop a line from F through I meeting line BC at G (Figure # 28, p. 192). Here we find the two half-pentalphas.

It has certainly been recognized before that the pentalpha is basic to the construction of the pentagon, and hence the dodecahedron. Thus, Cornford remarks: "[the Dodecahedron] was in fact constructed by means of an isosceles triangle having each of its base angles double of the vertical angle" (Cornford, 1956, p. 218). In conjunction with Plato's two primitive triangles, and the equilateral triangle, representing the monad, odd, and even, the pentalpha (or half-pentalpha) adds the fourth triangle which completes the tetraktys. The _Timaeus_ opens with Socrates saying: "One, two, three, but where my dear Timaeus, is the fourth of those who were yesterday my guests and are to be my entertainers today" (_Timaeus_ 17a)? Of course, Critias, Timaeus of Locri, and Hermocrates of Syracuse are present. But the unnamed guest is absent due to indisposition. However, I think this is also a cryptic suggestion to discover the fourth triangle. This I would argue is the half-pentalpha or pentalpha.

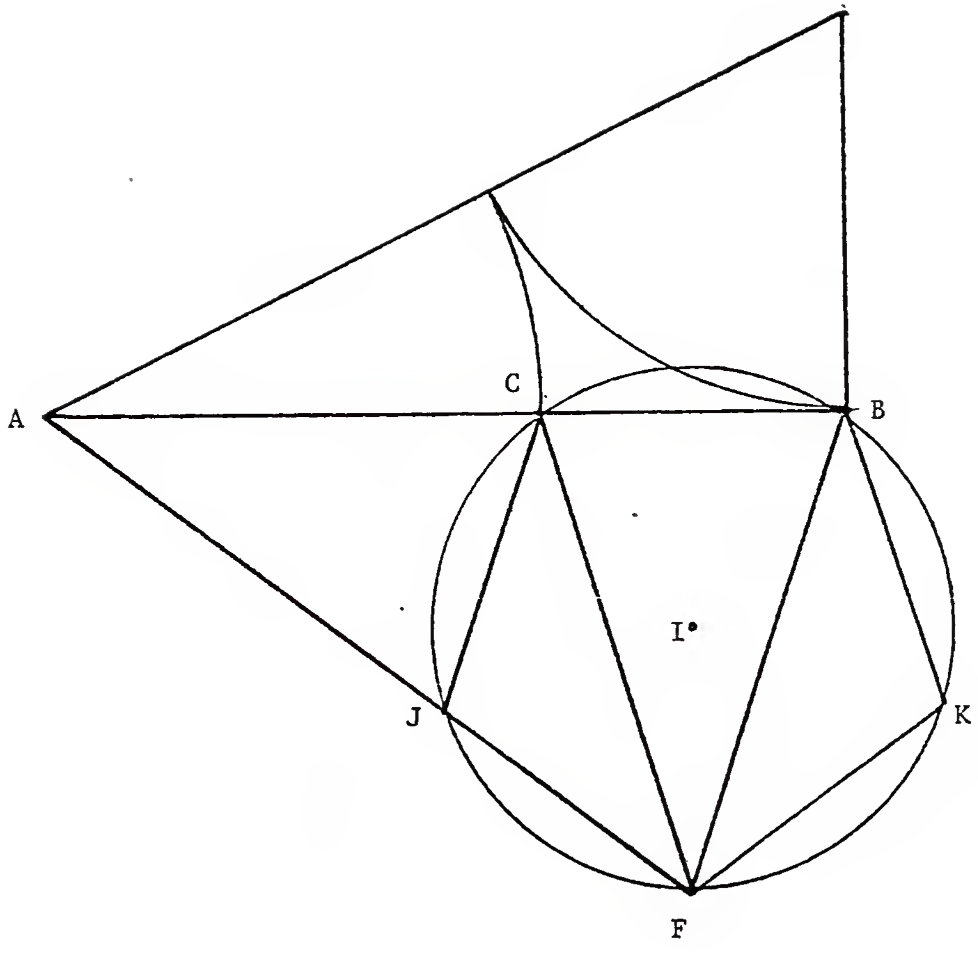


Figure # 26: Pentagon in Circle

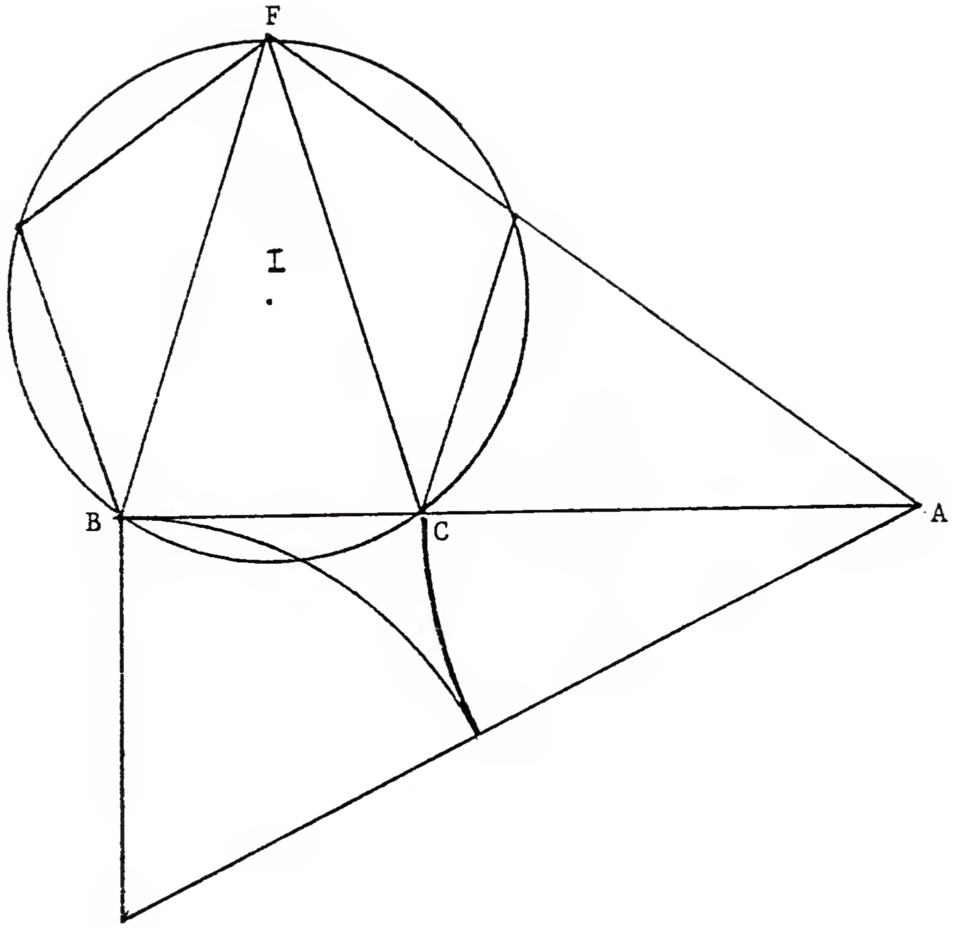


Figure # 27: 180° Rotation of Figure # 26

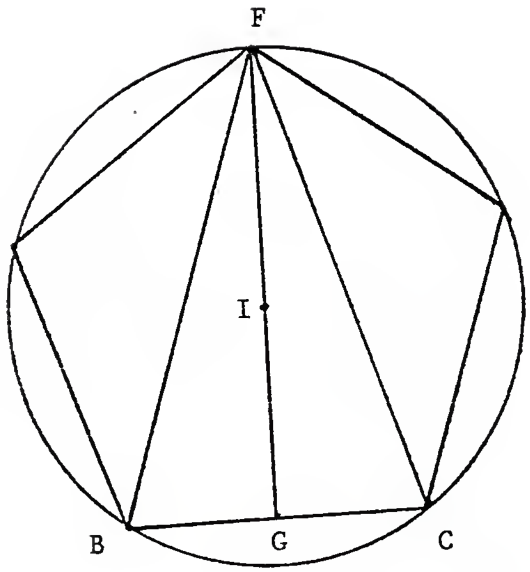


Figure # 28: Circle, Pentagon, & Half-Pentalphas

Now if we take the pentagram BKFJC inscribed in a circle (see Figure # 29, p. 194), we find the following properties:

KJ = ϕ^3
KL = ϕ
LM = 1
IS/IR = $\phi/2$
IK/IS = 2ϕ
If T is the intersection point of two diagonals MR and LP, then:
PT/TL = ϕ
RT/TM = ϕ
and FT/TQ = ϕ.
IF PL produced meets KF in N, since NLP is parallel to KC, then:
FN/NK = ϕ
FL/LR = ϕ
FP/PL = ϕ.

What then can be said in general about the transformability of the regular solids? Or was Vlastos correct in supposing the transformability breaks down? To answer these questions let us first construct the golden rectangle. For this we will consider Figure #'s 30 , 31, & 32, p. 195. Let us begin with the construction of the rectangle ABCD, which is composed of two squares, ABFE and EFCD. Next let us divide one of these squares at GH. Then with compass at center H and radius HF, draw an arc from F meeting ED in I. Then draw the line IJ parallel to CD. The resulting rectangle ABJI is a golden rectangle. If JI is 1, then AI is ϕ. Also the smaller rectangle EFJI is a golden rectangle. Now a gnomon is the smallest surface which can be added to a given surface to produce a similar surface. One of the features of the golden rectangle is that it is the only rectangle of which the gnomon is a

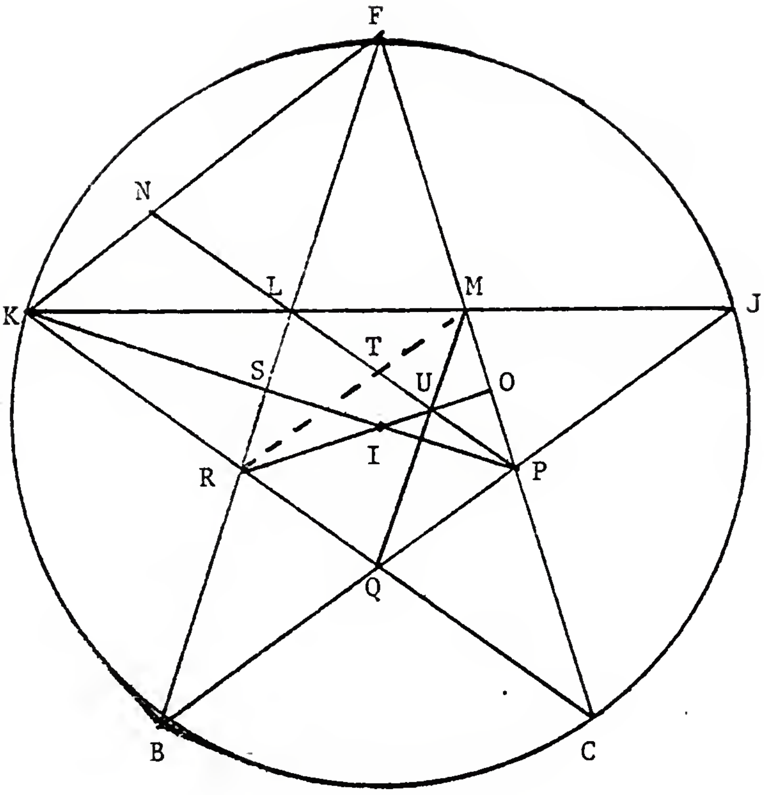


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Figure # 29: Golden Section in Pentagram

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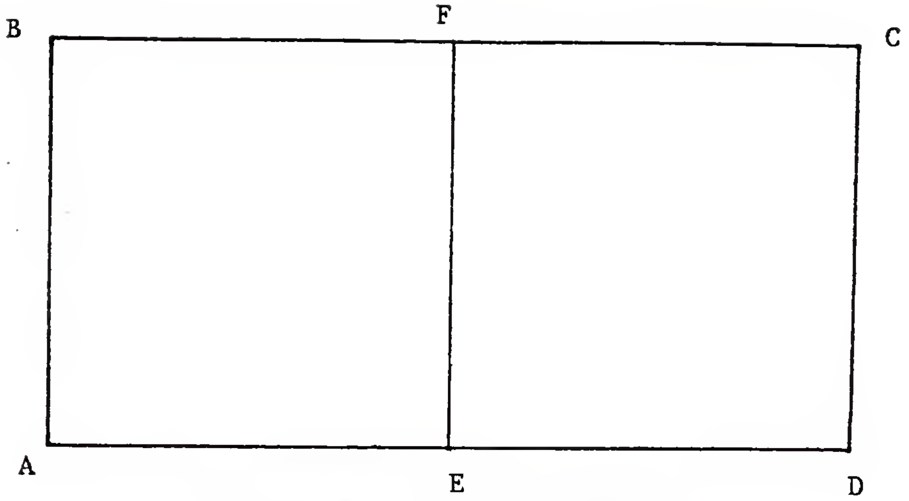

Figure # 30: Double Square

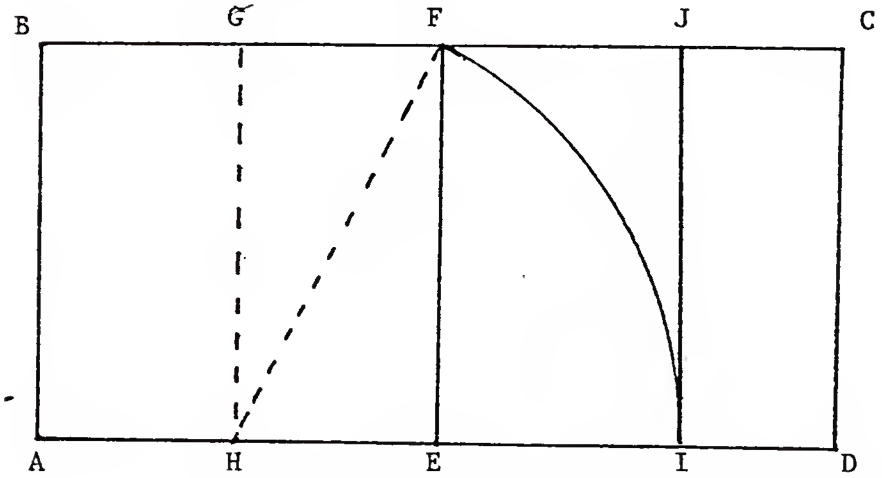

Figure # 31: Construction of Golden Rectangle

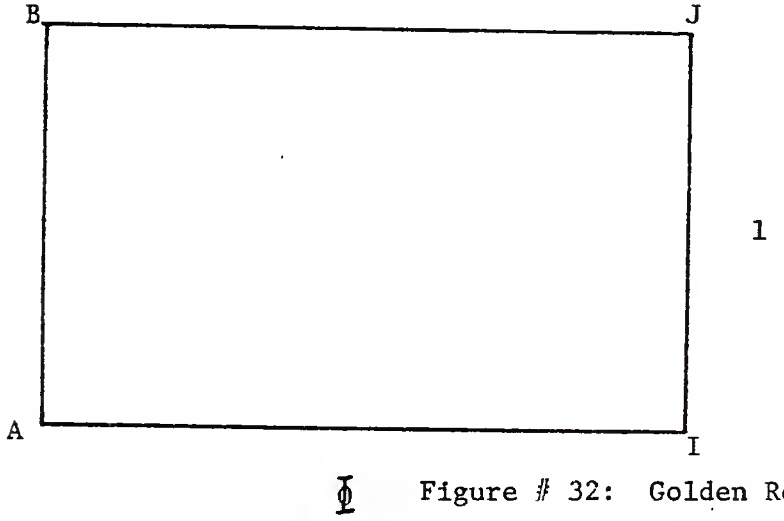

Figure # 32: Golden Rectangle

square. Like the pentalpha or golden triangle, the golden rectangle is another example of the embodiment of the golden ratio in two dimensions. As Ghyka writes,

> This most "logical" asymmetrical division of a line, or of a surface, is also the most satisfactory to the eye; this has been tested by Fechner (in 1876) for the "golden rectangle," for which the ratio between the longer and the shorter side is ϕ = 1.618.... In a sort of "Gallup Poll" asking a great number of participants to choose the most (aestheticall) pleasant rectangle, this golden rectangle or ϕ rectangle obtained the great majority of votes. (Ghyka, 1946, pp. 9-10)

These golden rectangles are very important in considering the transformability and relations of the five regular solids.

> . . . two pairs of the Platonic solids are reciprocal and the fifth is self-reciprocating in this sense: if the face centers of the cube are joined, an octahedron is formed, while the joins of the centroids of the octahedron surfaces form a cube. Similar relationship holds between the icosahedron and the dodecahedron. The join of the four centroids of the tetrahedron's faces makes another tetrahedron. . . . The twelve vertices of a regular icosahedron are divisible into three coplanar groups of four. These lie at the corners of three golden rectangles which are symmetrically situated with respect to each other, being mutually perpendicular, their one common point being the centroid of the icosahedron [see Figure # 33, p. 197]. An icosahedron can be inscribed in an octahedron so that each vertex of the former divides an edge of the latter in th golden section. The centroids of the twelve pentagonal faces of a dodecahedron are divisible into three coplanar groups of four. These quadrads lie at the corners of three mutually perpendicular, symmetrically placed golden rectangles, their one common point being the centroid of the dodecahedron [see Figure # 34, p. 197]. (Huntley, 1970, pp. 33-34)

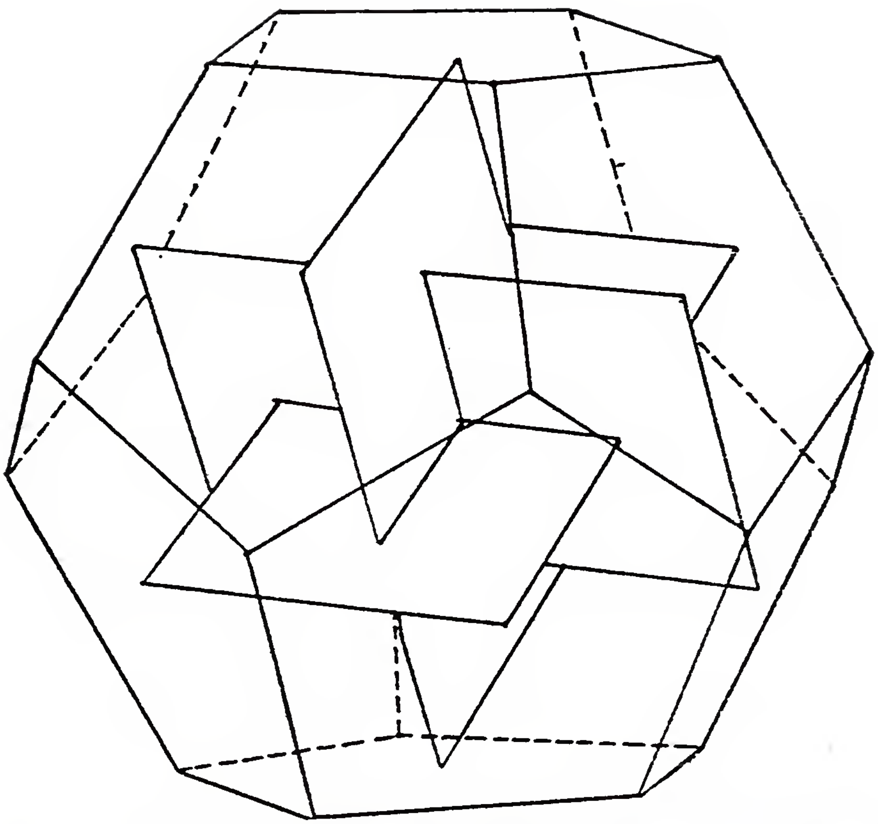

Figure # 34: Dodecahedron with Intersecting Golden Rectangles

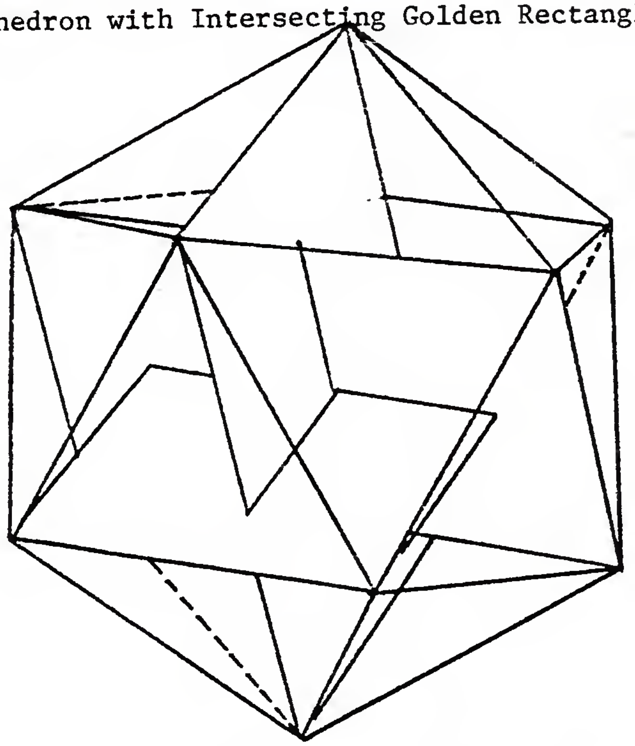

Figure # 33: Icosahedron with Intersecting Golden Rectangles

Consider for a moment Figures # 33 and # 34, p. 197. The former is an icosahedron embodying three intersecting golden rectangles. The latter figure is a dodecahedron, embodying the same intersecting golden rectangles. These are the propotional interrelations that are relevant in Plato's cosmos. Thus, the five regular solids are transformable in terms of their mutually reciprocating properties. The relation of each to one another when inscribed within a sphere brings to the forefront the predominance of the fundamental invariant, the divine ratio, the golden section. It is the fundamental invariant transmitted into the cosmos by the world soul.

This is how number, ratio, and proportion get infused into the world. According to Plato, prior to this the world is not separated into elements. Only when proportion is imposed upon it do we get the elements and various qualities. Thus Plato says,

> . . . when all things were in disorder, God created in each thing in relation to itself, and in all things in relation to each other, all the measures and harmonies which they could possibly receive. For in those days nothing had proportion except by accident, nor was there anything deserving to be called by the names which we now use--as, for example, fire, water, and the rest of the elements. All these [the elements] the creator first set in order, and out of them he constructed the universe. . . . (Timaeus 69b-c)

My own view is that the relevance of the golden section to the regular solids can be most easily seen by examining Euclid's Book XIII of the Elements. Heath

maintains that the first five propositions of this Book are due to Eudoxus. He says,

> it will be remembered that, according to Proclus, Eudoxus "greatly added to the number of theorems which originated with Plato regarding the section" (i.e., presumably the "golden section"); and it is therefore probable that the five theorems are due to Eudoxus. (Heath, 1956, vol. 3, p. 441)

The five propositions are in order as follows:

> I. If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square on the half.
> II. If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in exteme and mean ratio, the greater segment is the remaining part of the original straight line.
> III. If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment.
> IV. If a straight line be cut in extreme and mean ratio, the square on the whole and square on the lesser segment together are triple of the square on the greater segment.
> V. If a straight line be cut in exteme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line has been cut in extreme and mean ratio, and the orginal straight line is the greater segment. (Heath, 1956, vol. 3, pp. 440-451)

These first 5 propositions are only preparatory for the later propositions, especially propositions 17 and 18.

> XVII. To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational line called apotome.
> XVIII. To set out the sides of the five figures and to compare them with one another.

It is very interesting that the first five propositions of Book XIII contain alternative proofs using an ancient method of analysis. As Heath remarks,

> Heiberg (after Bretschneider) suggested in his edition (Vol. v. p. lxxxiv.) that it might be a relic of analytical investigations by Theaetetus or Eudoxus, and he cited the remark of Pappus (v. p. 410) at the beginning of his "comparisons of the five [regular solid] figures which have an equal surface," to the effect that he will not use "the so-called analytical investigation by means of which some of the ancients effected their demonstrations."(Heath, 1956, vol. 3, p. 442)

The proof follows the abduction pattern that I have suggested Plato employed. The analysis part of the proof begins by assuming what is to be proved and then moves backwards to an evident hypothesis. Then in the synthesis, the direction is reversed. And beginning with the result which was arrived at last in the analysis, one moves downwards deductively, deriving as a conclusion that which was set out initially to be proven. It seems that this is a remnant of the analytic or abductive investigations carried on by Plato and the members of the Academy. The proof is even more intriguing because not only does it contain a remnant of the analytic method, but it also involves the golden section. When the analysis of the regular solids is carried to the first dimension of lines, and thence to the ratio of number, we discover that it is the golden section that is fundamental for Plato. The method of analysis and the golden section appear to mutually complement one another. They are, I contend, at the center of Plato's philosophy.

Conclusion

In this dissertation I have accomplished three things. First, through argument and example, I have exposed the plethora of evidence that Plato was primarily a Pythagorean, concerning himself, and the members of his Academy, with the discovery and application of mathematical principles to philosophy. Secondly, I have traced the roots of Perircean abduction back to Plato's (and his pupils') use of an ancient method of analysis. This "anapalin lusin," or reasoning backwards, was employed in the Academy to solve problems, and to provide rational explanations for anomalous phenomena. And thirdly, through an examination of the evidence for Plato's "intermediate mathematicals," the role of proportion, and the interplay of the One (Good) and Indefinite Dyad, I have set forth the premiere importance of the golden section (τομη) for Plato.

I also accomplished the task I set for myself in the Introduction (supra p.5&6). That is, I set out to find the "more beautiful form than ours for the construction of these [regular] bodies" (Timaeus 54a). I discovered the fourth triangle in the half-pentalpha. It completes the tetraktys. I suggest that the monad then is the equilateral triangle. The dyad is Plato's right-angled isosceles triangle, the half-square. The triad is Plato's

right-angled scalene triangle, the half-equilateral. And I have added the tetrad as the half-pentalpha.

But the most beautiful form is even more fundamental than this. When one takes the fifth solid, the dodecahedron reserved for the whole, and reduces it from the third to the second dimension, the result is either the plane figure of the golden rectangle or the golden triangle (pentalpha). The further reduction to the first dimension yields the line divided in mean and extreme ratio. And finally, the reduction to number yields the relation of ϕ (the golden section) to the One (unit or monad). It is, therefore, the golden section which is the most beautiful form. And we discovered it embodied in the Republic's Divided Line, in the continued geometrical proportion $\phi^2 : \phi :: \phi : 1 :: 1 : 1/\phi$.

Throughout this dissertation I have employed the very method of analysis (or abduction) that I have attributed to Plato. Consistent with this, I have contended that Plato is obstetric in his dialogues. Accordingly, a good abduction must make what was previously anomalous, the expected. It must, so to speak, normalize the paranormal. The overall hypothesis (and the lesser hypotheses) that I have abducted, provide a basis for making many of the Platonic riddles and puzzles easily understandable. My position goes far to eliminate the mystery surrounding Plato's identification of numbers with Forms, the references to intermediate mathematicals, the open question

of participation in the Academy, the nature of the unwritten lectures, the superficial bifurcation of the Forms and the sensible world, and the discrepancies between the dialogues and Aristotle's account of Plato's philosophy. Also, the suggestions that Plato discovered analysis, taught it to Leodamos, and started the theorems regarding the golden section (which Eudoxus then multiplied using analysis), become much more reasonable.

Many of the elements I have woven together in this dissertation have been touched upon by others before me. I have but managed to tie together some loose ends, and provide a consistent overall framework for Plato's philosophy. I only hope that enough groundwork has been set for another to penetrate even more deeply into the Platonic arcanum. So much has been written on Plato, and yet we have only begun to dip beneath the veil to view the real essence of his work.

Notes

Plato's termination of the analysis seems too obvious to overlook. It is relevant to consider here a passage by Aristotle:

> Empedocles declares that it (soul) is formed out of all the elements, each of them also being soul; in the same way Plato in the Timaeus fashions the soul out of his elements. . . . Similarly also in his lectures "On Philosophy": it was set forth that the Animal-itself is compounded of the Idea itself of the One together with the primary length, breadth and depth, everything else, the objects of its perception, being similarly constituted. Again he puts his view in yet other terms: Mind is the monad, science or knowledge the dyad (because it

> goes undeviatingly from one point to another), opinion the number of the plane (the triad), sensation the number of the solid (the tetrad). (*DeAnima* 404b12-24)

Why then does Plato stop with the triad or opinion? Just as one ascends the Divided Line of the *Republic,* so one should ascend this analogous scale. The triangle should be analyzed into lines, and from thence into numbers and their ratios.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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August 1983

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